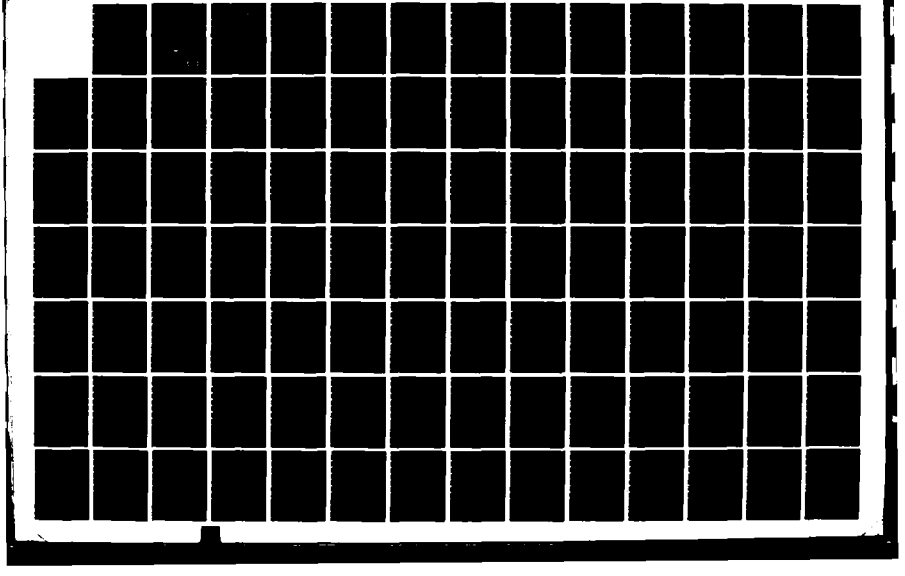
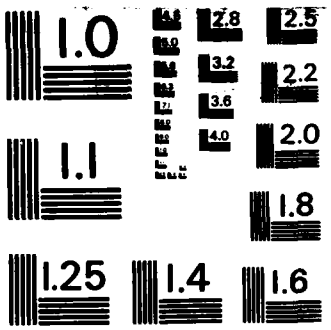
AD-A139 929 NUMERICAL SOLUTION OF ALGEBRAIC MATRIX RICCATI EQUATIONS(U) NAVAL WEAPONS CENTER CHINA LAKE CA W F ARNOLD FEB 84 NWC-TP-6521 SBI-AD-E900 331 1/2

UNCLASSIFIED F/G 12/1 NL





MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

NWC TP 6521

(12)

Numerical Solution of Algebraic Matrix Riccati Equations

by
William Fredrick Arnold III
Electronic Warfare Department

FEBRUARY 1984

NAVAL WEAPONS CENTER
CHINA LAKE, CALIFORNIA 93555

Naval Weapons Center
AN ACTIVITY OF THE NAVAL MATERIAL COMMAND

FOREWORD

This report documents and distributes a dissertation submitted by the author in partial satisfaction of the requirements for the degree of Doctor of Philosophy in Electrical Engineering at the University of Southern California, Los Angeles.

The research was conducted during the period September 1981 to August 1983.

Approved by
P. B. HOMER, *Head*
Electronic Warfare Department

Under authority of
K. A. DICKERSON
Capt., U.S. Navy
Commander

Released for publication by
B. W. HAYS
Technical Director
February 1984

NWC Technical Publication 6521

Published by ..... Electronic Warfare Department
Collation ..... Cover, 83 leaves
First printing ..... 95 unnumbered copies

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE *(When Data Entered)*

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER NWC TP 6521	2. GOVT ACCESSION NO. AD-A139 929	3. RECIPIENT'S CATALOG NUMBER
4. TITLE *(and Subtitle)* Numerical Solution of Algebraic Matrix Riccati Equations		5. TYPE OF REPORT & PERIOD COVERED Final 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR*(s)* William Fredrick Arnold III		8. CONTRACT OR GRANT NUMBER*(s)*
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Weapons Center China Lake, California 93555		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Naval Weapons Center China Lake, California 93555		12. REPORT DATE February 1984 13. NUMBER OF PAGES 163
14. MONITORING AGENCY NAME & ADDRESS*(if different from Controlling Office)*		15. SECURITY CLASS. *(of this report)* UNCLASSIFIED 15a. DECLASSIFICATION DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT *(of this Report)*

Approved for public release; distribution is unlimited.

17. DISTRIBUTION STATEMENT *(of the abstract entered in Block 20, if different from Report)*

18. SUPPLEMENTARY NOTES

19. KEY WORDS *(Continue on reverse side if necessary and identify by block number)*

Riccati Equations	Optimal Control	Balancing	Second-Order Models
Generalized Matrix Riccati Equation	Kalman Filtering	Generalized Eigenproblem	Controllability
Discrete Riccati Equation	Numerical Condition	Numerical Software	Stabilizability
Descriptor Systems	Newton Iteration	FORTRAN Software	Observability
			Detectability

20. ABSTRACT *(Continue on reverse side if necessary and identify by block number)*

See back of form.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE
S/N 0102-LF-014-6601

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE *(When Data Entered)*

(U) *Numerical Solution of Algebraic Matrix Riccati Equations* (U), by William Fredrick Arnold III. China Lake, Calif., Naval Weapons Center, February 1984. 163 pp. (NWC TP 6521, publication UNCLASSIFIED.)

(U) Numerical issues related to the computational solution of the algebraic matrix Riccati equation are studied. The approach uses the generalized eigenproblem formulation for the solution of general forms of algebraic Riccati equations arising in both continuous- and discrete-time applications. These general forms result from control and filtering problems for systems in generalized (or implicit or descriptor) state space form. A Newton-type iterative refinement procedure for the generalized Riccati solution is derived. The issue of numerical condition of the Riccati problem is addressed. Balancing to improve numerical condition is discussed. An overview of a software package coded in FORTRAN is given. Results of numerical experiments are reported.

(U) The special structure of models of physical systems given in linear second-order form is examined. Exploiting the structure in solving associated Riccati equations is studied. Tests for controllability and observability are derived in terms of the original second-order-model matrices.

CONTENTS

Figures:

Tables:

CHAPTER 1

INTRODUCTION

In this study some numerical issues related to the computational solution of a generalized form of the algebraic matrix Riccati equation are examined. The approach herein utilizes the generalized eigenproblem formulation, which provides a powerful framework for the solution of quite general forms of algebraic Riccati equations arising in both continuous- and discrete-time applications. This general form is derived from control and filtering problems for systems in generalized (or implicit or descriptor) state space form. These equations play fundamental roles in the analysis, synthesis, and design of linear-quadratic-Gaussian control and estimation systems as well as in other areas of applied mathematics. A representative sample of applications may be found in [1]-[4]. It is not our purpose here to survey the extensive literature available on Riccati equations, but, rather we refer the reader to, for example [1]-[4] for references.

The method exploited here is a variant of the classical eigenvector approach to Riccati equations. Martensson [5] is one of the best summaries of the eigenvector approach to solving algebraic Riccati equations. However, the use of eigenvectors often encounters numerical difficulties in practical computation, especially when the correspond ing eigenvalues are closely spaced. Thus, the method to be preferred and the basis of this work employ Schur vectors instead of eigenvectors because Schur vectors are more reliably and accurately computed. The Schur method was first examined in detail by Laub [6], [7] and

then extended by Pappas, et.al. [8] and Emami-Naeini [9] for the discrete-time problem, and by Van Dooren [10] for the continuous- and discrete-time problem. The very general formulation herein was derived by Laub [11] and expanded by Lee [12].

The numerical issue first examined in this thesis is the derivation of a Newton-type iterative-refinement procedure for the generalized problem formulation. Kleinman [13] and Hewer [14] derived methods of this type for the standard Riccati equation in the continuous- and discrete-time characterizations, respectively.

Perhaps the most important issue studied here is that of conditioning of the Riccati problem, that is, to define a measure of the sensitivity of the numerical solution to changes in the data. Although results are available on conditioning for problems such as linear equations [15], [16] and eigenvalues [17]-[19], little is published on the conditioning of the Riccati problem. A recent work in this area for the continuous-time problem is by Byers [20]. The results of numerical experiments examining various condition measures are reported herein.

Since the behavior of many physical systems in engineering is first modeled by second-order differential equations with very special properties (i.e., symmetry and definiteness), we examine exploiting this structure in certain computations of interest. Namely, we exploit the structure in solving associated Riccati problems and in testing for controllability and observability of the second-order models.

1.1 Main Contributions of the Thesis

We regard the derivation of condition measures for the continuous- and discrete-time generalized algebraic Riccati equation and the evaluation of their numerical behavior in special situations as the main contributions of this thesis. More specifically, we may list the following contributions.

1. The derivation of a Newton-type iterative-refinement procedure for the continuous and discrete algebraic Riccati equation that has monotonic convergence that is quadratic in the vicinity of the solution.
2. Establishing the equivalence between the given generalized optimal control problem solution and the solution of a more standard optimal control problem in certain cases.
3. Derivation of condition numbers for the solution of the generalized Riccati equation.
4. A new algorithm for applying a change of coordinate transformation to the system model when the Riccati problem is solved using the generalized eigenproblem approach.
5. Numerical experimentation to illustrate the behavior of known and speculated condition estimates for the Riccati problem. The numerical results demonstrate that a single satisfactory condition estimate is not available.
6. Exploitation of the structure of second-order models to solve efficiently the velocity feedback optimal control problem using a generalized Riccati equation.

7. Formulation of useful tests for controllability and observability of second-order models directly in terms of conditions on the model matrices.
8. Development of a portable FORTRAN software package for the efficient, reliable solution of generalized algebraic Riccati equations.

1.2 Outline of Chapters

In Chapter 2 we review some important concepts of numerical analysis, which are used heavily in the succeeding analysis. The concepts of numerical stability and conditioning are presented, and there is a brief discussion of first-order perturbation analysis. The application of Schur techniques for the solution of generalized algebraic Riccati equations is reviewed. Very general optimal control and filtering problems are formulated, which result in generalized algebraic Riccati equations for the continuous- and discrete-time cases.

Chapter 3 derives a Newton-type iterative procedure, which we employ for improving the numerical accuracy of the Schur solution of generalized Riccati equations or for calculating new solutions when problem parameters are changed by a small amount. Newton's method for the standard algebraic Riccati equation in continuous- and discrete-time formulations is reviewed first. Equivalence between the generalized solution and a certain standard solution is established. The Newton procedure is then extended to the generalized case.

The subject of conditioning of the generalized algebraic Riccati equation is examined in Chapter 4. The desired form of condition

estimate is defined. Previous work in the area of conditioning of the Riccati problem is reviewed. New condition estimates are then derived using a first-order perturbation analysis. Balancing to improve the condition of the problem is next considered. Balancing of the generalized eigenproblem is discussed first. A new algorithm for applying a change of coordinates transformation to the system model when the Riccati problem is solved in the generalized eigenproblem formulation is presented.

Numerical results for the solution of the generalized Riccati equation for special cases are presented in Chapter 5. The software package developed as a research tool to aid in the studies of this thesis is discussed. Highlights of the package capabilities are given. The numerical algorithms employed and sources for the software are discussed and referenced. The results of three specially designed examples are examined to illustrate the behavior of the Riccati solution and the ability of the known and new condition estimates to predict the numerical accuracy of the solution.

In Chapter 6 second-order models are considered. The second-order model structure is defined in the large space structure framework. The generalized algebraic Riccati equation is considered for this problem, and ways are explored to take computational advantage of the second-order formulation to solve the Riccati equation. The velocity feedback problem is shown to reduce to a simple form in this framework. Tests for controllability (stabilizability) and observability (detectability)

are then derived in terms of the model matrices for second order models.

Finally in Chapter 7 we state the conclusions we have drawn from this work and make recommendations for future research in this area.

An appendix is included which contains the description (from the actual software) of all the FORTRAN subroutines accumulated in the software package.

CHAPTER 2

BACKGROUND

In this chapter we review some important concepts and results which form the basis for the work in the succeeding chapters. The first section discusses the numerical analysis concepts of stability and conditioning. The second section briefly presents the perturbation theory necessary in later analysis. The third section reviews results employing Schur techniques for the solution of algebraic matrix Riccati equations. Very general optimal control and filtering problems are formulated which result in generalized algebraic Riccati equations (GARE) for the continuous- and discrete-time cases. Solutions for these GARE are stated in terms of a generalized eigenproblem.

2.1 Numerical Stability and Conditioning

We shall now introduce the important concepts of numerical stability and conditioning. The following framework will be useful for the understanding of these concepts:

$$\underset{\substack{\text{problem}\\ \text{or}\\ \text{model}}}{f} : \underset{\text{data}}{D} \longrightarrow \underset{\text{solutions}}{S} \qquad (2.1)$$

Give $d \in D$, we want to compute $f(d) \in S$. However, frequently only d^* (near d) is known and only f^* (an algorithm to approximate f) is available. Figure 2.1 is a geometrical view of this problem-solving process. Therefore, in this framework we seek $f(d)$, but compute $f^*(d^*)$.

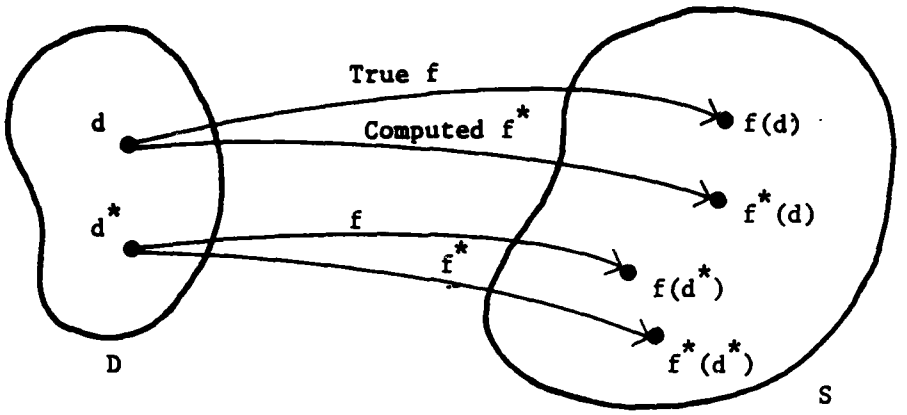


FIGURE 2.1 Geometric view of the effect of data uncertainty and computational errors in solving a problem.

2.1.1 Numerical Stability

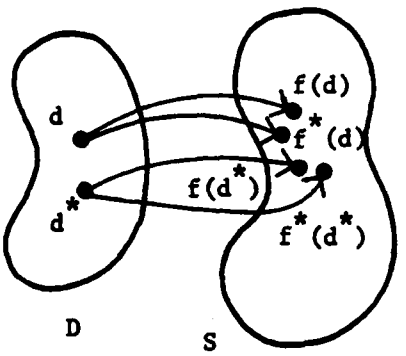
Definition 2.1: An algorithm f* is numerically stable if for all d contained in D, there exists a d* contained in D and near d such that f(d*) (the exact solution of a slightly perturbed problem) is near f*(d) (the computed solution).

That is, we expect that a stable algorithm will not introduce any inaccuracies into the solution larger than those present in the data.

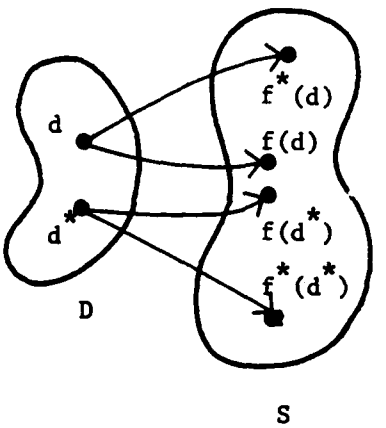
2.1.2 Problem Conditioning

Definition 2.2: If f(d*) (the exact solution of a slightly perturbed problem) is near f(d) (the true solution), the problem is said to be well-conditioned. If f(d*) may potentially differ greatly from f(d), the problem is ill-conditioned.

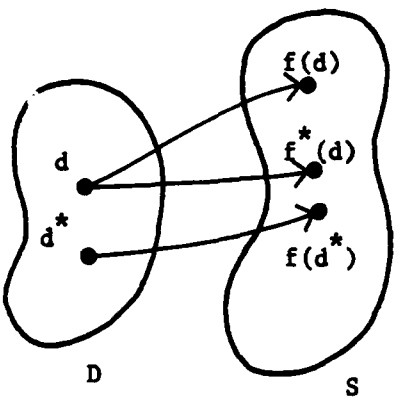
The four possible combinations of stability and conditioning are shown in Figure 2.2.



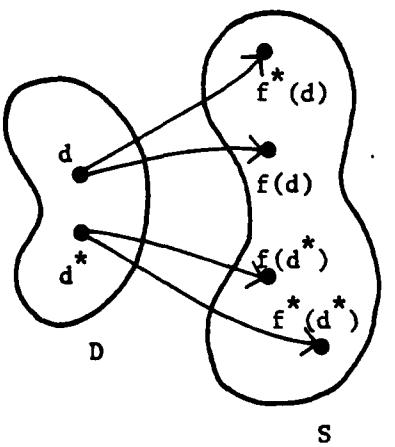
a) Well-conditioned problem
Stable computation



b) Well-conditioned problem
Unstable computation



c) Ill-conditioned problem
Stable computation



d) Ill-conditioned problem
Unstable computation

FIGURE 2.2 Possible combinations of problem conditioning and computation stability in the numerical solution of a problem.

Although it may be impossible for the numerical solution process to guarantee an accurate answer to an ill-conditioned problem, it is desirable for the program to recognize ill-conditioning of the computations and report this fact to the user. Therefore, it is desirable to associate data with a computing problem which reflects the overall sensitivity of the solution to changes in the data (i.e., a condition number). We can do this as follows [21]:

Definition 2.3: Let D and S be finite dimensional metric spaces with metrics ρ_d and ρ_s, respectively. Let f(d) be the computing problem under consideration. As before

$$f : D \longrightarrow S$$

The absolute asymptotic condition number is

$$\kappa\big(f(d)\big) := \lim_{\delta \to 0} \sup_{\rho_d(d,d^*)=\delta} \frac{\rho_s\big(f(d),f(d^*)\big)}{\delta} \tag{2.2}$$

Relative condition can be defined similarly [21]. Figure 2.3 illustrates the concept of a condition number. One can see that κ is a measure of the sensitivity of the solution to perturbations in the data. κ can be interpreted as the factor by which the uncertainty, δ, is multiplied by in the problem-solving process. Condition numbers are generally estimated using perturbation theory.

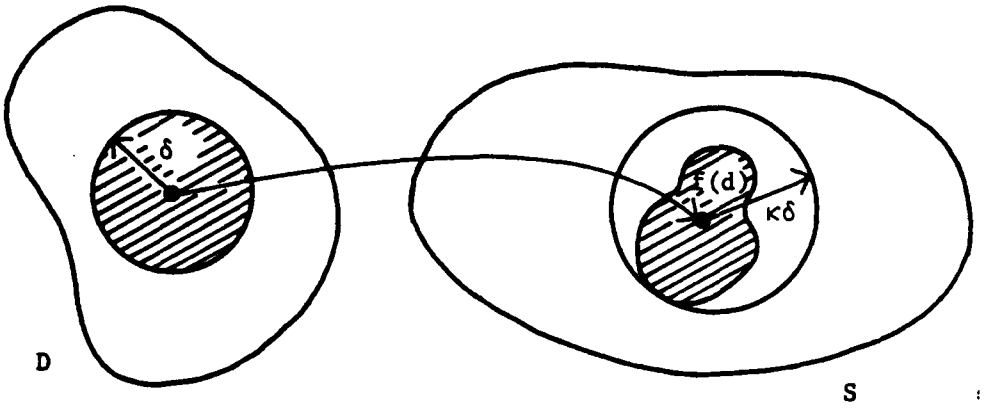


FIGURE 2.3 Geometric interpretation of the condition number, κ.

2.1.3 Condition Number of a Matrix

Definition 2.4: Let $\|\cdot\|$ be a consistent matrix norm on $R^{n\times n}$ satisfying

$$\|I\| = 1 \tag{2.3}$$

with a consistent vector norm on R^n. Let $A \in R^{n\times n}$, then

$$\kappa(A) = \|A\| \, \|A^{-1}\| \quad . \tag{2.4}$$

Now consider the computation of the inverse of a slightly perturbed matrix, i.e., find $(A + E)^{-1}$.

Theorem 2.1: If $\|A^{-1}\| \cdot \|E\| < 1$, then

$$\frac{\|A^{-1}-(A+E)^{-1}\|}{\|A^{-1}\|} \leq \frac{\kappa(A)\,\frac{\|E\|}{\|A\|}}{1-\kappa(A)\frac{\|E\|}{\|A\|}} \tag{2.5}$$

Proof: [15]

The left hand side of the inequality is the relative error in $(A + E)^{-1}$. If E is sufficiently small, the right side is effectively $\kappa(A)\,\|E\|/\|A\|$. Therefore, the theorem states that the relative error in A + E may be magnified by as much as $\kappa(A)$ in calculating $(A + E)^{-1}$. For this reason, $\kappa(A)$ is called "the condition number of A with respect to inversion." Note that

$$1 = \|I\| = \|AA^{-1}\| \leq \|A\|\,\|A^{-1}\| = \kappa(A) \tag{2.6}$$

2.1.4 Error Analysis

In the numerical solution of real problems, we compute $f^*(d^*)$ instead of f(d), so we would like to estimate $\|f(d) - f^*(d^*)\|$. This is the basic goal of error analysis. There are two main types of error analysis of numerical computation and are referred to as forward and backward error analysis.

In the forward error analysis, one attempts to obtain a bound on the error in the final result by starting with the original problem and following, step by step, the effect of computational errors (round-off) and original data uncertainty. However, the resulting bounds are usually hopelessly pessimistic, and the analysis itself is very complicated for all but simple problems [16], [22].

In backward error analysis, one does not attempt to compute $\|f(d) - f^*(d^*)\|$, but rather one attempts to determine how close the problem actually solved is to the original problem. This is illustrated graphically in Figure 2.4. In this technique, one uses error bounds to

show that the computed solution of a given problem is near the exact solution of a slightly perturbed problem. This is sufficient to ensure that the algorithm that did the computations is stable.

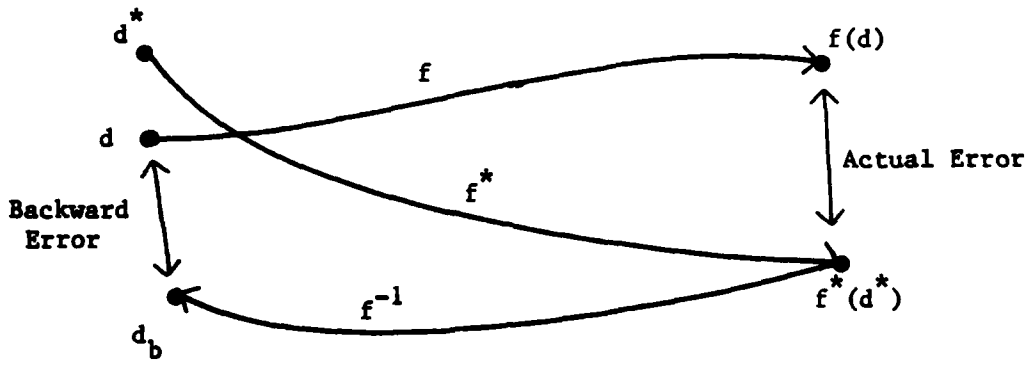


FIGURE 2.4 Illustration of backward error.

2.1.5 Role of Orthogonal Matrices

The class of orthogonal transformations has a special role in numerical computations.

Definition 2.5: An orthogonal matrix U is a square matrix with orthonormal columns. That is

$$U \in R^{n \times n},\ U = (u_1, u_2, \ldots, u_n)$$

and

$$u_i^T u_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

Properties of orthogonal matrices important to numerical computations are

Theorem 2.2: Let U be orthogonal, then

$$\text{1) } U^TU = UU^T = I \text{ ;} \qquad \text{2) } \|AU\|_2 = \|UA\|_2 = \|A\|_2. \tag{2.7}$$

Proof: 1) $U^TU = I$ follows from the orthonormality of the columns of U. It follows then that U^T is the inverse of U. Since by definition a matrix commutes with its inverse, we have $UU^T = I$. Note that this implies that the rows of U are also orthonormal.

2) To prove this part, we make use of the facts that for $A \in R^{mxn}$, $\|A\|_2^2 = \|A^TA\|_2$ and $\|A^T\|_2 = \|A\|_2$ ([15], Theorem 4.2.10). Now,

$$\|UA\|_2^2 = \|(UA)^TUA\|_2 = \|A^TU^TUA\|_2 = \|A^TA\|_2 = \|A\|_2^2$$

and, therefore,

$$\|AU\|_2 = \|(AU)^T\|_2 = \|U^TA^T\|_2 = \|A^T\|_2 = \|A\|_2.$$

From this it is easy to see that orthogonal matrices have three advantages. First, they are easy to invert; $U^{-1} = U^T$. Second, orthogonal matrices are perfectly conditioned with respect to $\|\cdot\|_2$:

$$\kappa(U) = \|U\|_2\|U^T\|_2 = 1 \cdot 1 = 1. \tag{2.8}$$

A third advantage of orthogonal transformations is that perturbations in the result can be accounted for by a perturbation of the same size in the original problem. For example, having computed U^TAU, we introduce an error F into the result. If we set $E = UFU^T$, then $\|E\|_2 = \|F\|_2$ and

$$U^T(A + E)U = U^TAU + F \ . \tag{2.9}$$

This makes orthogonal transformations ideal for backward error analysis. That is, multiplications by orthogonal transformations are backward stable.

2.2 Perturbation Analysis

For the purposes of this thesis, we employ a technique commonly known as first-order-perturbation theory. This technique reveals the effects of perturbations in the problem on the solution. The technique is generally applied in a three-step procedure. First, a form is chosen for the approximate problem solution. Then the approximation is substituted into an equation, and all terms second-order or higher in small quantities are deleted. Finally, the resulting linear equation is solved for the unknown in the approximation.

To illustrate the technique, consider the problem of estimating $(A + E)^{-1}$, where A is nonsingular and E is a "small-perturbation" matrix. First we chose a form for the approximation as

$$(A + E)^{-1} \approx A^{-1}(I - H), \tag{2.10}$$

where H is a "small perturbation" matrix. Substituting the approximation into the equation

$$(A + E)(A + E)^{-1} = I, \tag{2.11}$$

we obtain

$$(A + E)A^{-1}(I - H) = I - H + EA^{-1} - EA^{-1}H \approx I . \tag{2.12}$$

Dropping the term $EA^{-1}H$, which is assumed small in comparison to E and H, and solving the resulting equation gives us

$$H = EA^{-1}, \tag{2.13}$$

or

$$(A + E)^{-1} \simeq A^{-1} (I - EA^{-1}). \tag{2.14}$$

We can now loosely derive the condition number of a matrix with respect to inversion. From (2.14) we have

$$\|(A + E)^{-1} - A^{-1}\| \simeq \|A^{-1}EA^{-1}\| \leq \|A^{-1}\|^2\|E\| \tag{2.15}$$

Hence, we have

$$\frac{\|(A+E)^{-1}-A^{-1}\|}{\|A^{-1}\|} \leq \|A^{-1}\|\|E\| = \kappa(A) \frac{\|E\|}{\|A\|} . \tag{2.16}$$

This differs from the bound of Theorem 2.1 (equation 2.5) by the factor

$$\frac{1}{1-\kappa(A)\frac{\|E\|}{\|A\|}} \tag{2.17}$$

which is negligible when $\|E\|$ is small.

2.3 Solution of the Generalized Algebraic Riccati Equation (GARE) as a Generalized Eigenproblem

In this section we shall present the method of solving the GARE (both continuous- and discrete-time cases) via a generalized eigenproblem. First, the GARE resulting from the optimal regulator problem will be derived. Then the optimal filtering problem will also be stated, and the corresponding GARE given. Finally, the solutions to the GARE will be formulated utilizing the generalized real Schur form of the generalized eigenproblem.

2.3.1 Optimal Regulator - Continuous-Time Problem

Consider the following general time-invariant deterministic linear optimal regulator problem:

$$\text{System:} \quad E\dot{x}(t) = Ax(t) + Bu(t) \quad ; x(t_0) = x_0 \tag{2.18}$$

$$y(t) = Cx(t)$$

$$\text{Criterion:}\quad J = \tfrac{1}{2}\int_0^\infty (y^T Q y + u^T R u + 2\, x^T S u)\,dt \tag{2.19}$$

where

$$x \in R^n;\ u \in R^m;\ y \in R^r; \tag{2.20}$$

$$E, A \in R^{n\times n};\ B \in R^{n\times m};\ C \in R^{r\times n},$$

Also, $\quad Q = Q^T \in R^{r\times r};\ R = R^T \in R^{m\times m};\ S \in R^{n\times m};$

and

$$\begin{bmatrix} C^T Q C & S \\ S^T & R \end{bmatrix} \geq 0\ ;\quad Q, R > 0\ \ ;\ E \text{ nonsingular.}$$

Application of Hamilton-Jacobi theory gives rise to the following equations:

$$h = \tfrac{1}{2}\,(x^T C Q C x + 2x^T S u + u^T R u) + p^T(Ax + Bu - E\dot{x}) \tag{2.22}$$

$$\frac{\partial h}{\partial p} - \frac{d}{dt}\left\{\frac{\partial h}{\partial \dot{p}}\right\} = 0 = Ax + Bu - E\dot{x} \tag{2.23}$$

$$\frac{\partial h}{\partial x} - \frac{d}{dt}\left\{\frac{\partial h}{\partial \dot{x}}\right\} = 0 = C^T Q C x + Su + A^T p + E^T\dot{p} \tag{2.24}$$

$$\frac{\partial h}{\partial u} - \frac{d}{dt}\left\{\frac{\partial h}{\partial \dot{u}}\right\} = 0 = S^T x + Ru + B^T p \tag{2.25}$$

where h is the scalar Hamiltonian, and p(t) is the costate vector, $p \in R^n$.

In the usual treatment E = I and S = 0. equation (2.25) is solved for u, and u is then substituted in (2.23) and (2.24) to obtain the Hamiltonian system

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -C^T Q C & -A^T \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} \tag{2.26}$$

If one makes the "Riccati substitution" p = Xx, we are led to the algebraic Riccati equation for X

$$A^T X + XA - XBR^{-1}B^T X + C^T QC = 0 \tag{2.27}$$

If the pair (A,B) is stabilizable and the pair (C,A) is detectable, then a unique non-negative definite solution $X = X^T \geq 0$ exists such that the linear control law

$$u^0 = -R^{-1}B^T x \tag{2.28}$$

stabilizes the system and is optimal in the sense that it minimizes the criterion (2.19) over all other control laws [1].

In the more general setting we have the following Hamiltonian system

$$\begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} A-BR^{-1}S^T & -BR^{-1}B^T \\ SR^{-1}S^T-C^TQC & -(A-BR^{-1}S^T)^T \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} \tag{2.29}$$

If one makes the "Riccati substitution" $p = XEx$, we are led to the GARE

$$\begin{aligned} 0 &= (A-BR^{-1}S^T)^T XE + E^T X(A-BR^{-1}S^T) - E^T XBR^{-1}B^T XE + C^T QC - SR^{-1}S^T \\ &= A^T XE + E^T XA - (B^T XE + S^T)^T R^{-1} (B^T XE + S^T) + C^T QC \end{aligned} \tag{2.30}$$

and the optimal control law is given by

$$u^0 = -R^{-1}(S^T + B^T XE)x \quad . \tag{2.31}$$

To solve for $X = X^T \geq 0$, one can use the technique suggested by Pappas, et.al. [8] for the discrete time ARE as expanded by Lee [12]. Before presenting this technique, we will formulate the other problems of interest.

2.3.2 Optimal Filter - Continuous-Time Problem

Consider the following time-invariant linear optimal observer problem

$$\text{System:}\quad E\dot{x}(t) = Ax(t) + Bu(t) + Dv(t), \tag{2.32}$$

$$y(t) = Cx(t) + w(t),\ t \geq t_0,$$

where $\begin{bmatrix} v(t) \\ w(t) \end{bmatrix} \in R^{\ell+r}$ is a white noise process with intensity

$$\begin{matrix} & \ell & r \\ \ell & \\ r & \end{matrix} \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} > 0 \tag{2.33}$$

Also, (2.20) applies, E is nonsingular and $D \in R^{n\times\ell}$. Furthermore, the initial state $x(t_0)$ is uncorrelated with v and w, and u(t) is a given input to the system.

The optimal observer is given by

$$E\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K^0[y(t) - C\hat{x}(t)]\ , \tag{2.34}$$

which minimizes the weighted mean square reconstruction error

$$E\{e^T(t)We(t)\}\ , \tag{2.35}$$

where

$$e(t) = x(t) - \hat{x}(t);\ W = W^T > 0\ . \tag{2.36}$$

Let

$$\bar{S} = ES\ , \tag{2.37}$$

Then K^0 is given by

$$K^0 = [\bar{S} + EXC^T]\ R^{-1}\ , \tag{2.38}$$

where X solves the following GARE:

$$0 = (A-\bar{S}R^{-1}C)XE^T + EX(A-\bar{S}R^{-1}C)^T - EXC^TR^{-1}CXE^T + DQD^T - \bar{S}R^{-1}\bar{S}^T. \tag{2.39}$$

One can see that this GARE is the dual to (2.30), and, therefore, we can solve for X by the same technique.

2.3.3 Optimal Regulator - Discrete-Time Problem

Consider the following general discrete-time, time-invariant deterministic optimal regulator problem:

$$\text{System:} \quad Ex_{k+1} = Ax_k + Bu_k \; ; \; x(0) = x_0 \tag{2.40}$$

$$y_k = Cx_k$$

$$\text{Criterion:} \quad J = \frac{1}{2} \sum_{k=0}^{\infty} (y_k^T Q y_k + u_k^T R u_k + 2x_k^T S u_k) \; . \tag{2.41}$$

where (2.20) and (2.21) apply also except we only require $R \geq 0$. Application of the discrete maximum principle [23] gives rise to the set of equations

$$h_k = \frac{1}{2} \; x_k^T C^T Q C x_k + u_k^T R u_k + 2x_k^T S u_k \; + p_{k+1}^T \; Ax_k + Bu_k \tag{2.42}$$

$$E^T p_k = \frac{\partial h_k}{\partial x_k} = C^T Q C x_k + S u_k + A^T p_{k+1} \tag{2.43}$$

$$\frac{\partial h_k}{\partial u_k} = 0 = R u_k + S^T x_k + B^T p_{k+1} \tag{2.44}$$

$$E\lambda_{k+1} = \frac{\partial h_k}{\partial p_{k+1}} = Ax_k + Bu_k \tag{2.45}$$

where h_k is the scalar Hamiltonian, and $p_k \in R^n$ is the costate vector.

Assuming for the moment that R is invertible, we can solve (2.44) for u_k and substitute into (2.43) and (2.45). Also, we make the "Riccati substitution" $p_k = XEx_k$ to obtain the following discrete version of the GARE (for nonsingular X):

$$E^TXE = (A - BR^{-1}S^T)^T(X^{-1} + BR^{-1}B^T)^{-1}(A - BR^{-1}S^T)$$

$$+ C^TQC - SR^{-1}S^T \quad . \tag{2.46}$$

This equation can be algebraically manipulated into the following equivalent forms:

$$E^TXE = (A - BR^{-1}S^T)^T X(A - BR^{-1}S^T) - (A - BR^{-1}S^T)^T XB(R$$

$$+ B^TXB)^{-1}B^TX(A - BR^{-1}S^T) + C^TQC - SR^{-1}S^T \tag{2.47}$$

$$= A^TXA - (A^TXB + S)(R + B^TXB)^{-1}(A^TXB + S)^T + C^TQC. \tag{2.48}$$

Note that (2.47) does not require X^{-1} explicitly, and (2.48) does not require X^{-1} or R^{-1} explicitly.

Once X is determined from the above GARE, the linear optimal feedback is given by

$$u_k^0 = -(R + B^TXB)^{-1}(A^TXB + S)^T x_k \quad . \tag{2.49}$$

As in the continuous-time case, stabilizability and detectability assumptions assure the existence and uniqueness of $X = X^T \geq 0$ such that (2.49) is a stabilizing feedback.

2.3.4 Optimal Filter - Discrete-Time Problem

Consider the following discrete-time, time-invariant linear optimal observer problem

$$\text{System:} \quad Ex_{k+1} = Ax_k + Bu_k + Dv_k$$

$$y_k = Cx_k + w_k \qquad k > 0 \tag{2.50}$$

where $v_k \in R^{\ell}$ and $w_k \in R^r$ are zero mean sequences with

$$\mathbb{E}\left\{\begin{bmatrix} v_k \\ w_k \end{bmatrix}\begin{bmatrix} v_\ell^T & w_\ell^T \end{bmatrix}\right\} = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}\delta_{k\ell} \tag{2.51}$$

$$Q = Q^T \in R^{\ell x \ell}, \quad S \in R^{\ell x r}, \quad R = R^T \in R^{r x r} .$$

Equation (2.20) applies, E is nonsingular and $D \in R^{n x \ell}$. The input sequence u_k is known, and the initial state x_0 is uncorrelated with v_k and w_k.

It can be shown that the optimal observer is given by

$$E\hat{x}_{k+1} = A\hat{x}_k + Bu_k + K^0(y_k - C\hat{x}_k) . \tag{2.52}$$

The weighted mean square reconstruction error

$$\mathbb{E}\{(x_k - \hat{x}_k)^T W(x_k - \hat{x}_k)\} , \tag{2.53}$$

where $W > 0$, is minimized when K is chosen to be

$$K^0 = (AXC^T + DS)(CXC^T + R)^{-1} \tag{2.54}$$

and X is found by solving the GARE:

$$EXE^T = AXA^T + DQD^T - (AXC^T + DS)(CXC^T + R)^{-1}(AXC^T + DS)^T \tag{2.55}$$

or equivalently

$$EXE^T = (A - DSR^{-1}C)X(A - DSR^{-1}C)^T + DQD^T - DSR^{-1}S^TD^T$$
$$- (A - DSR^{-1}C)XC^T(CXC^T+R)^{-1}CX\,(A - DSR^{-1}C)^T \tag{2.56}$$

Equations (2.55) and (2.56) are dual to (2.48) and (2.47), and we can solve either problem using the same technique. We shall now present a technique for solving the GARE.

2.3.5 GARE Solution - Continuous-Time Case

We will work in the context of the regulator problem for convenience. If we define

$$y = \begin{bmatrix} x \\ p \end{bmatrix} ; \; L = \begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix} \; ; \; M = \begin{bmatrix} A-BR^{-1}S^T & -BR^{-1}B^T \\ SR^{-1}S^T-C^TQC & -(A-BR^{-1}S^T)^T \end{bmatrix} ; \quad (2.57)$$

Then we can associate with (2.29) a generalized eigenproblem

$$\lambda Ly = My. \quad (2.58)$$

Theorem 2.3: (Generalized Real Schur Form) let L, M ϵ R^{2nx2n} and define the matrix pencil $\lambda L - M$. The pencil is said to be regular if $\det(\lambda L - M) \not\equiv 0$. There exist orthogonal transformations P and Z such that

$$P^T(\lambda L - M)Z = \lambda P^T LZ - P^T MZ = \lambda \hat{L} - \hat{M} \quad (2.59)$$

where $\hat{L}$ is upper triangular and $\hat{M}$ is quasi-upper triangular. For the 1x1 diagonal blocks, the generalized eigenvalues are real $\lambda_i = \hat{m}_{ii}/\hat{\ell}_{ii}$ (possibly "infinite" if $\hat{\ell}_{ii} = 0$). The 2x2 blocks correspond to a finite pair of complex conjugate eigenvalues. Moreover, these eigenvalues can be arranged in any desired order.

Proof: [10]

It can be shown for L and M, as defined in (2.57), that if λ is a generalized eigenvalue of $\lambda L - M$, then $-\lambda$ is also (Hamiltonian property of the eigenvalues). If the transformed pencil $P^T(\lambda L - M)Z$ is partitioned as

$$P^T(\lambda L - M)Z = \lambda \begin{bmatrix} L_{11} & L_{12} \\ 0 & L_{22} \end{bmatrix} - \begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{bmatrix} \tag{2.60}$$

where L_{11}, $M_{11} \in R^{nxn}$, then the vectors corresponding to the first n columns of Z span the eigenspace of $\lambda L_{11} - M_{11}$ [10]. Moreover, we can require that the real parts of the generalized eigenvalues of $\lambda L_{11} - M_{11}$ be negative. If we partition the corresponding Z into nxn blocks as

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} , \tag{2.61}$$

we can state the following:

Theorem 2.4: With respect to the assumptions made for this generalized eigenvalue formulation with L, M and Z as defined above,

1) Let $V = \begin{bmatrix} E & 0 \\ 0 & I \end{bmatrix} Z$ then $X = V_{21} V_{11}^{-1}$ (2.62)

solves (2.30) with $X = X^T \geq 0$;

2) The generalized eigenvalues of $\lambda L_{11} - M_{11}$ are the closed-loop eigenvalues of the system under the optimal feedback

$$u^0(t) = -R^{-1}(S^T + B^T X E)x(t)$$

Proof: [12]

The above method depends explicitly on R^{-1} and can encounter computational difficulties when R is ill-conditioned with respect to inversion. If this is the case, the following compression technique due to Van Dooren [10] can be employed. Arrange equations (2.23) through (2.25) as

$$\begin{bmatrix} E & 0 & 0 \\ 0 & E^T & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{p} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} A & 0 & B \\ -C^TQC & -A^T & -S \\ S^T & B^T & R \end{bmatrix} \begin{bmatrix} x \\ p \\ u \end{bmatrix} . \tag{2.63}$$

Determine an orthogonal matrix $U \in R^{(2n+m)x(2n+m)}$ such that

$$\left[\begin{array}{c|c} U_{11} & U_{12} \\ \hline U_{21} & U_{22} \end{array}\right] \left[\begin{array}{c} B \\ \hline -S \\ \hline R \end{array}\right] = \left[\begin{array}{c} 0 \\ \hline 0 \\ \hline \bar{R} \end{array}\right] , \tag{2.64}$$

where $\bar{R} \in R^{mxm}$ and is nonsingular. Then apply U to the pencil corresponding to (2.63)

$$U \left[\lambda \begin{bmatrix} E & 0 & 0 \\ 0 & E^T & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} A & 0 & B \\ -C^TQC & -A^T & -S \\ S^T & B^T & R \end{bmatrix} \right] \tag{2.65}$$

to obtain the pencil

$$\lambda U_{11} \begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix} - \left[U_{11} \begin{bmatrix} A & 0 \\ -C^TQC & -A^T \end{bmatrix} + U_{12} \begin{bmatrix} S^T & B^T \end{bmatrix} \right]. \tag{2.66}$$

It can be shown [10] that the pencils (2.66) and (2.58) are equivalent. Therefore, the pencil (2.66) can be used to solve for X without having to invert R.

2.3.6 GARE Solution - Discrete-Time Case

As in the previous section, we work in the context of the regulator problem for convenience. The matrix pencil appropriate for this problem can be formed from equations (2.43) through (2.45) as

$$\lambda \begin{bmatrix} E & 0 & 0 \\ 0 & A^T & 0 \\ 0 & -B^T & 0 \end{bmatrix} - \begin{bmatrix} A & 0 & B \\ -C^TQC & E^T & -S \\ S^T & 0 & R \end{bmatrix} \tag{2.67}$$

Assuming for the moment that R is invertible, we can derive a pencil equivalent to (2.67) in the form $\lambda L - M$ as

$$\lambda \begin{bmatrix} E & BR^{-1}B^T \\ 0 & (A-BR^{-1}S^T)^T \end{bmatrix} - \begin{bmatrix} A-BR^{-1}S^T & 0 \\ -C^TQC+SR^{-1}S^T & E^T \end{bmatrix} \tag{2.68}$$

where L and M are defined appropriately. It can be shown that the generalized eigenvalues of (2.68) have the symplectic property, i.e., if λ is a generalized eigenvalue of (2.68), then $\frac{1}{\lambda}$ is also [12].

We can now solve the discrete-time problem in a manner analogous to the continuous-time case. We first transform the pencil (2.68) to the form (2.60) using orthogonal P and Z. Note if R is singular or ill-conditioned with respect to inversion, we can compress the pencil (2.67) to a pencil equivalent to (2.68) using the technique of Van Dooren [10]. Now, we will require that $|\lambda|$ corresponding to the resulting pencil $\lambda L_{11} - M_{11}$ be less than unity. If we partition Z as in (2.61), we have

Theorem 2.5: With respect to the assumptions made for this generalized eigenvalue formulation with L, M, and Z as defined above,

1) Let $V = \begin{bmatrix} E & 0 \\ 0 & I \end{bmatrix} Z$ then

$$X = V_{21}\, V_{11}^{-1} \tag{2.69}$$

solves (2.48) with $X = X^T \geq 0$;

2) The generalized eigenvalues of $\lambda L_{11} - M_{11}$ are the closed-loop eigenvalues under the optimal feedback

$$u^0_k = -(R + B^T X B)^{-1}(S^T + B^T X A)x_k$$

Proof: [12]

Note that in this discrete-time problem, neither the solution of the GARE nor the optimal feedback depends explicitly on R^{-1} as in the continuous-time problem. The special case R = 0 is called "deadbeat control" and is discussed for the case S = 0, E = I, utilizing the above methods for solution in [24].

CHAPTER 3

ITERATIVE REFINEMENT BY NEWTON'S METHOD

Numerical implementation of the GARE solution methods of Chapter 2 is relatively straightforward. Proven stable algorithms exist that are coded into reliable FORTRAN software (see Chapter 5). However, the GARE may be ill-conditioned, and the resulting numerical solution may not be as accurate as desired. This chapter presents an iterative refinement procedure utilizing Newton's method. This procedure is presented for the continuous- and discrete-time problems. The continuous-time method is based on Kleinman [13], and the discrete-time method is based on Hewer [14].

3.1 Iterative Refinement of Continuous-Time Solution

The following result is due to Kleinman [13] for the controllable case, which was extended to the stabilizable case by Sandell [25]. This result is for the ARE and is presented here in detail, although from a slightly different approach, to provide the groundwork for extension to the GARE.

Given the system (2.18) and criterion (2.19) with E=I, Q=I and S=0, the resulting ARE from (2.27) is

$$A^TX + XA - XBR^{-1}B^TX + C^TC = Q. \tag{3.1}$$

If the pair $[A,B]$ is stabilizable and the pair $[C,A]$ is detectable, then there exists a unique solution $X=X^T \geq 0$ to (3.1), such that the linear feedback

$$u^0 = -R^{-1}B^T X x \quad := -Kx \tag{3.2}$$

stabilizes the closed-loop system and minimizes the criterion (2.19). Furthermore, it can be shown that

$$J(x_0;u^0) = \min_u J(x_0;u) = x_0^T X x_0 . \tag{3.3}$$

To derive the Newton iterative procedure, consider at the k-th iteration that the solution X of (3.1) is of the form

$$X = X_k + \delta X , \tag{3.4}$$

where δX is "small", i.e., can neglect second-order terms in δX. Substituting (3.4) into (3.1), we have

$$0 = A^T X_k + X_k A - X_k B R^{-1} B^T X_k + C^T C + \delta X (A - B R^{-1} B^T X_k) + (A - B R^{-1} B^T X_k)^T \delta X . \tag{3.5}$$

Denote the solution of (3.5) at the k-th iteration by δX_k, and let

$$\delta X_k = X_{k+1} - X_k \quad . \tag{3.6}$$

Substituting (3.6) into (3.5) and simplifying, we have

$$0 = (A - BK_{k+1})^T X_{k+1} + X_{k+1}(A - BK_{k+1}) + C^T C + K_{k+1}^T R K_{k+1} , \tag{3.7}$$

where

$$K_{k+1} = R^{-1} B^T X_k \quad , \tag{3.8}$$

and $A - BK_{k+1}$ is the closed-loop-system matrix.

Equation (3.7) is a Lyapunov equation in the unknown X_{k+1} and its unique non-negative definite solution is given by

$$X_{k+1} = \int_0^\infty e^{(A-BK_{k+1})^T t} \; (C^T C + K_{k+1}^T R K_{k+1}) e^{\; (A-BK_{k+1})t} dt, \quad (3.9)$$

which is finite, if and only if, the closed-loop-system matrix is stable (i.e., has eigenvalues with negative real parts). It can be shown that

$$X_1 - X_2 = \int_0^\infty e^{(A-BK_2)^T t} [(K_1-K_2)^T R(K_1-K_2) - (K_1-K_2)^T (B^T X_1 - RK_2)$$
$$- (B^T X_1 - RK_2)^T (K_1-K_2)] e^{(A-BK_2)t} dt \quad (3.10)$$

or, alternatively,

$$X_1 - X_2 = \int_0^\infty e^{(A-BK_1)^T t} [(K_1-K_2)^T R(K_1-K_2) - (K_1-K_2)^T (B^T X_2 - RK_2)$$
$$- (B^T X_2 - RK_2)^T (K_1-K_2)] e^{(A-BK_1)t} \; dt. \quad (3.11)$$

We can now state and prove the main result of Kleinman [13] as extended by Sandell [25].

Theorem 3.1: Let X_k, k=0,1,..., be the unique non-negative definite solution of the linear algebraic equation

$$0 = (A-BK_k)^T X_k + X_k (A-BK_k) + C^T C + K_k^T R K_k \quad (3.12)$$

where, recursively,

$$K_k = R^{-1} B^T X_{k-1} \; , \; k = 1,2,..., \quad (3.13)$$

and K_0 is chosen such that the matrix $(A-BK_0)$ is stable. Then,

1) $0 \leq X \leq X_{k+1} \leq X_k \leq \ldots \leq X_0$

2) $\lim_{k\to\infty} X_k = X$

3) in the vicinity of X, $\|X_{k+1}-X\| \leq C_2 \|X_k-X\|^2$,

where C_2 is a finite constant.

Proof: 1) Let X_0 satisfy (3.12) for the chosen K_0. Now set $K_1 = R^{-1}B^T X_0$ and let X_1 be the associated solution to (3.9). Using (3.10), we obtain

$$X_0 - X_1 = \int_0^\infty e^{(A-BK_1)^T t}(K_0-K_1)^T R(K_0-K_1)e^{(A-BK_1)t}\, dt \geq 0 ,$$

so that $X_1 \leq X_0$. In addition, we have by (3.11)

$$X_1 - X = \int_0^\infty e^{(A-BK_1)^T t}(K_1-K)^T R(K_1-K)e^{(A-BK_1)t}\, dt \geq 0.$$

Hence, X_1 is bounded above and below and, therefore, has finite norm. Thus, $(A-BK_1)$ is stable so X_1 satisfies (3.12) with k=1. Repeating the above argument for k=2,3,... yields the desired result.

2) Taking the limit of (3.12) as $k\to\infty$, we obtain

$$0 = A^T X_\infty + X_\infty A - X_\infty BR^{-1}B^T X_\infty + C^T C . \qquad (3.14)$$

Since X is the unique non-negative definite solution of (3.14), $X_\infty = X$.

3) Set $K_1 = R^{-1}B^T X_k$ and $K_2 = R^{-1}B^T X$ in (3.11) and take the norm to find

$$\|X_{k+1} - X\| \leq \int_0^\infty \|e^{(A-BK_{k+1})\tau}\|^2 d\tau \; \|BR^{-1}B^T\| \; \|X_k - X\|^2 . \qquad (3.15)$$

Since $(A-BK_{k+1})$ is stable, it can be shown that, uniformly in k,

$$\int_0^\infty \|e^{(A-BK_{k+1})\tau}\|^2 d\tau \leq \text{constant} = C_1$$

Let $C_2 = C_1\|BR^{-1}B^T\|$, and the proof is complete.

It should be noted that another iterative scheme is possible and has been used in practice [26]. One simply solves (3.5) at iteration k for δX_k and then takes

$$X_{k+1} = X_k + \delta X_k \ . \tag{3.16}$$

It has been shown [27] that

$$\ell im_{k\to\infty} \delta X_k = 0 \tag{3.17}$$

and

$$\ell im_{k\to\infty} X_k = X \ . \tag{3.18}$$

This procedure, while theoretically equivalent to Theorem 3.1, has some computational drawbacks. Namely, it requires more machine operations and if the initial guess for X_0 is nonsymmetric, then X_{k+1} will be nonsymmetric at each iteration.

Before we extend Theorem 3.1 to the general case, we first want to show that solving the GARE (2.27), which results from the generalized state space system (2.18) with criterion (2.19) and then applying the optimal feedback (2.31), is equivalent to solving a "standard" regulator problem when E is invertible.

We will always assume that E is invertible since when E is singular the GARE (2.27) does not, in general, have a solution. However, even though E is assumed nonsingular, it is computationally undesirable in many cases to invert E and solve the equivalent standard problem. A method for solving the generalized regulator problem when E is singular has been proposed by Cobb [28] and involves decomposing the problem into a standard part and a nonsingular E part.

Theorem 3.2: Solving the generalized regulator problem of section 2.3.1 is equivalent to solving the following "standard" problem

$$\dot{x} = \bar{A}x + \bar{B}u$$

$$y = Cx, \tag{3.19}$$

with criterion

$$J = \frac{1}{2}\int_0^\infty (y^T Q y + 2x^T S u + u^T R u)dt \tag{3.20}$$

where

$$\bar{A} = E^{-1}A$$

$$\bar{B} = E^{-1}B. \tag{3.21}$$

Proof: The ARE corresponding to this problem is

$$\begin{aligned} 0 &= (\bar{A}-\bar{B}R^{-1}S^T)^T\bar{X} + \bar{X}(\bar{A}-\bar{B}R^{-1}S^T) - \bar{X}\bar{B}R^{-1}\bar{B}^T\bar{X} + C^TQC - SR^{-1}S^T \\ &= (A-BR^{-1}S^T)^TE^{-T}\bar{X} + \bar{X}E^{-1}(A-BR^{-1}S^T) - \bar{X}E^{-1}BR^{-1}B^TE^{-T}\bar{X} \\ &\quad + C^TQC - SR^{-1}S^T. \end{aligned} \tag{3.22}$$

The optimal control law is

$$\begin{aligned} u^0 &= -R^{-1}(\bar{B}^T\bar{X} + S^T)x \\ &= -R^{-1}(B^TE^{-T}\bar{X} + S^T)x, \end{aligned} \tag{3.23}$$

which results in the closed-loop system

$$\dot{x} = E^{-1}(A-BR^{-1}S^T - BR^{-1}B^TE^{-T}\bar{X})x. \tag{3.24}$$

Now, if we note that

$$\bar{X} = E^TXE \tag{3.25}$$

then (3.22) is equivalent to (2.27), and (3.24) is equivalent to (2.18) with the feedback (2.31), that is,

$$\dot{Ex} = (A-BR^{-1}S^T - BR^{-1}B^TXE)x. \tag{3.26}$$

Now we will extend Theorem 3.1 to the general case.

Theorem 3.3: Let X_k, k=0,1,... be the unique non-negative definite solution of the linear algebraic equation

$$0 = (A-BK_k)^TX_kE + E^TX_k(A-BK_k) + C^TQC + K_k^TRK_k - SK_k - (SK_k)^T \tag{3.27}$$

where, recursively,

$$K_k = R^{-1}(B^TX_{k-1}E + S^T) \ , \ k=1,2,... \tag{3.28}$$

and where K_0 is chosen such that the matrix $E^{-1}(A-BK_0)$ is stable. Then

1) $0 \leq X \leq X_{k+1} \leq X_k \leq \ldots \leq X_0$

2) $\lim_{k\to\infty} X_k = X$

3) in the vicinity of X, $\|X_{k+1} - X\| \leq C_2\|X_k - X\|^2$

where X solves the GARE (2.27), and C_2 is a finite constant.

Proof: 1) We employ the results of Theorem 3.2 to convert the problem: from (3.25) let

$$X_k = E^{-T}\bar{X}_kE^{-1} \tag{3.29}$$

and substitute in (3.27) and (3.28) to obtain

$$0 = [E^{-1}(A-B\bar{K}_k)]^T\bar{X}_k + \bar{X}_kE^{-1}(A-B\bar{K}_k) + C^TQC + \bar{K}_k^TR\bar{K}_k - S\bar{K}_k - (S\bar{K}_k)^T, \tag{3.30}$$

where

$$\bar{K}_k = R^{-1}\{(E^{-1}B)^T\bar{X}_{k-1} + S^T\}. \tag{3.31}$$

Then

$$\bar{X}_k = \int_0^\infty e^{\left[E^{-1}(A-B\bar{K}_k)\right]^T t}\{C^TQC + \bar{K}_k^TR\bar{K}_k - S\bar{K}_k - (S\bar{K}_k)^T\}$$

$$e^{E^{-1}(A-B\bar{K}_k)t}dt \tag{3.32}$$

when $E^{-1}(A-B\bar{K}_k)$ is stable. It can be shown that

$$\bar{X}_1 - \bar{X}_2 = \int_0^\infty e^{\left[E^{-1}(A-B\bar{K}_2)\right]^T t}[(\bar{K}_1-\bar{K}_2)^TR(\bar{K}_1-\bar{K}_2)$$

$$- (\bar{K}_1-\bar{K}_2)^T\{(E^{-1}B)^T\bar{X}_1 - R\bar{K}_2 + S^T\}$$

$$- \{(E^{-1}B)^T\bar{X}_1 - R\bar{K}_2 + S^T\}^T(\bar{K}_1-\bar{K}_2)]e^{E^{-1}(A-B\bar{K}_2)t}dt \tag{3.33}$$

or, alternatively,

$$\bar{X}_1 - \bar{X}_2 = \int_0^\infty e^{\left[E^{-1}(A-B\bar{K}_1)\right]^T t}[(\bar{K}_1-\bar{K}_2)^TR(\bar{K}_1-\bar{K}_2)$$

$$- (\bar{K}_1-\bar{K}_2)^T\{(E^{-1}B)^T\bar{X}_2 - R\bar{K}_2 + S^T\}$$

$$- \{(E^{-1}B)^T\bar{X}_2 - R\bar{K}_2 + S^T\}^T(\bar{K}_1-\bar{K}_2)]e^{E^{-1}(A-B\bar{K}_1)t}dt\ . \tag{3.34}$$

Now let $\bar{X}_0$ satisfy (3.30) for a $\bar{K}_0$ chosen such that $E^{-1}(A-B\bar{K}_0)$ is stable. We remark here that by Theorem 3.2, this is equivalent to stabilizing $E^{-1}(A-BK_0)$. Set $\bar{K}_1 = R^{-1}((E^{-1}B)^TX_0 + S^T)$, and let $\bar{X}_1$ be the associated solution to (3.32). Using (3.33), we obtain

$$\bar{X}_0 - \bar{X}_1 = \int_0^\infty e^{\left[E^{-1}(A-B\bar{K}_1)\right]^T t}[(\bar{K}_0-\bar{K}_1)^TR(\bar{K}_0-\bar{K}_1)]e^{E^{-1}(A-B\bar{K}_1)t}dt > 0 \tag{3.35}$$

so that $\bar{X}_1 \leq \bar{X}_0$. In addition, we have by (3.34)

$$\bar{X}_1 - \bar{X} = \int_0^\infty e^{\left[E^{-1}(A-B\bar{K}_1)\right]^T t}\left[(\bar{K}_1-\bar{K})^T R(\bar{K}_1-\bar{K})\right]e^{E^{-1}(A-B\bar{K}_1)t}dt \geq 0. \tag{3.36}$$

Hence, $\bar{X}_1$ is bounded above and below and, therefore, has finite norm. Thus, $E^{-1}(A-B\bar{K}_1)$ is stable so $\bar{X}_1$ satisfies (3.30) with k=1. Repeating the above argument for k=2,3,... yields

$$0 \leq \bar{X} \leq \bar{X}_{k+1} \leq \bar{X}_k \leq \cdots \leq \bar{X}_0$$

Since by (3.29) E is a congruence transformation on X to yield $\bar{X}$, the desired result is implied.

2) Taking the limit of (3.27) as $k\to\infty$, we obtain after some manipulation

$$0 = (A-BR^{-1}S^T)^T X_\infty E + E^T X_\infty (A-BR^{-1}S^T) - E^T X_\infty BR^{-1}B^T X_\infty E + C^T QC - SR^{-1}S^T . \tag{3.37}$$

Since X is the unique non-negative definite solution to (3.37), $X_\infty = X$.

3) Set $\bar{K}_1 = R^{-1}\left[(E^{-1}B)^T\bar{X}_k + S^T\right]$ and $\bar{K}_2 = R^{-1}\left[(E^{-1}B)^T\bar{X} + S^T\right]$ in (3.34) to find

$$\bar{X}_{k+1} - \bar{X} = \int_0^\infty e^{\left[E^{-1}(A-B\bar{K}_{k+1})\right]^T t}\,(\bar{X}_k-\bar{X})E^{-1}BR^{-1}B^T\bar{E}^T (\bar{X}_k-\bar{X})e^{E^{-1}(A-B\bar{K}_{k+1})t}\,dt. \tag{3.38}$$

Substitute (3.25) into (3.38) to obtain

$$E^T(X_{k+1}-X)E = \int_0^\infty e^{\left[E^{-1}(A-BK_{k+1})\right]^T t} E^T(X_k-X)BR^{-1}B^T$$

$$(X_k-X)Ee^{E^{-1}(A-BK_{k+1})t}\,dt. \qquad (3.39)$$

Pre-multiply (3.39) by E^{-T}, post-multiply by E^{-1}, and take the norm to find

$$\|X_{k+1} - X\| \leq \int_0^\infty \|e^{E^{-1}(A-BK_{k+1})t}\|^2 dt\, \|E^{-1}\|^2 \|E\|^2 \|BR^{-1}B^T\| \|X_k-X\|^2. \qquad (3.40)$$

It can be shown that, uniformly in k,

$$\int_0^\infty \|e^{E^{-1}(A-BK_{k+1})t}\|^2 dt \leq \text{constant} = C_1.$$

Let $C_2 = C_1\kappa^2(E)\|BR^{-1}B^T\|$, and the proof is complete.

We remark that the proof to part 3 indicates that the square of the condition of E with respect to inversion, $\kappa^2(E)$, influences the convergence rate of the algorithm multiplicatively.

3.2 Iterative Refinement of Discrete-Time Solution

An iterative scheme is possible for the discrete-time formulation, and it was first reported by Hewer [14]. The result is for the discrete-time ARE, and the proof given here is similar to the continuous-time case.

Given the system (2.40) with criterion (2.41) with E=I, Q=I, and S=0, the resulting ARE from (2.48) is

$$X = A^TXA - A^TXB(R+B^TXB)^{-1}B^TXA + C^TC\ . \qquad (3.41)$$

If the pair $[A,B]$ is stabilizable and the pair $[C,A]$ is detectable, then there exists a unique solution $X = X^T \geq 0$ to (3.41) such that the linear feedback

$$u_k^0 = -(R+B^TXB)^{-1}B^TXAx_k := -Kx_k \tag{3.42}$$

stabilizes the closed-loop system and minimizes the criterion (2.41).

One can follow a first-order perturbation-type derivation to arrive at the Lyapunov equation that forms the basis of the following theorem in a manner similar to that of the previous section for the continuous-time problem. The following result is due to Hewer [14].

Theorem 3.4: Let X_k, $k=0,1,\ldots$, be the unique non-negative definite solution of the linear algebraic equation

$$X_k = (A-BK_k)^TX_k(A-BK_k) + K_k^TRK_k + C^TC, \tag{3.43}$$

where, recursively,

$$K_k = (R+B^TX_{k-1}B)^{-1}B^TX_{k-1}A, \quad k=1,2,\ldots, \tag{3.44}$$

and K_0 is chosen such that the closed-loop-system matrix $(A-BK_0)$ is stable (i.e., has eigenvalues whose magnitudes are less than unity). Then

1) $0 \leq X \leq X_{k+1} \leq X_k \leq \ldots \leq X_0$

2) $\lim_{k\to\infty} X_k = X$

3) in the vicinity of X, $\|X_{k+1} - X\| \leq C_2\|X_k - X\|^2$.

where C_2 is a finite constant.

Proof: 1) If $(A-BK_k)$ is stable, the unique non-negative definite solution X_k of (3.43) may be written as

$$X_k = \sum_{N=0}^{\infty} [(A-BK_k)^T]^N (K_k^T R K_k + C^T C)(A-BK_k)^N . \quad (3.45)$$

It can be shown that for $(A-BK_1)$ and $(A-BK_2)$ stable then

$$X_1 - X_2 = (A-BK_1)^T(X_1-X_2)(A-BK_1) + (K_1-K_2)^T(R+B^TX_2B)(K_1-K_2)$$
$$+ [(R+B^TX_2B)K_2 - B^TX_2A]^T(K_1-K_2) + (K_1-K_2)^T[(R+B^TX_2B)K_2$$
$$-B^TX_2A] \quad (3.46)$$
$$= \sum_{N=0}^{\infty} [(A-BK_1)^T]^N((K_1-K_2)^T(R+B^TX_2B)(K_1-K_2)+[(R+B^TX_2B)K_2$$
$$-B^TX_2A]+(K_1-K_2)^T[(R+B^TX_2B)K_2-B^TX_2A])(A-BK_1)^N, \quad (3.47)$$

or, alternatively,

$$X_1 - X_2 = (A-BK_2)^T(X_1-X_2)(A-BK_2) + (K_1-K_2)^T(R+B^TX_1B)(K_1-K_2)$$
$$+ [(R+B^TX_1B)K_2 - B^TX_1A]^T(K_1-K_2)+(K_1-K_2)^T[(R+B^TX_1B)K_2$$
$$-B^TX_1A] \quad (3.48)$$
$$= \sum_{N=0}^{\infty} [(A-BK_2)^T]^N((K_1-K_2)^T(R+B^TX_1B)(K_1-K_2)+[(R+B^TX_1B)K_2$$
$$-B^TX_1A]^T(K_1-K_2) + (K_1-K_2)^T[(R+B^TX_1B)K_2-B^TX_1A])$$
$$(A-BK_2)^N . \quad (3.49)$$

Let X_0 satisfy (3.43) for the chosen K_0. Now set $K_1 = (R+B^TX_0B)^{-1}$ B^TX_0A and let X_1 be the associated solution to (3.43). Using (3.49) we obtain

$$X_0 - X_1 = \sum_{N=0}^{\infty} [(A-BK_1)^T]^N[(K_0-K_1)^T(R+B^TX_0B)(K_0-K_1)](A-BK_1)^N \geq 0,$$

so that $X_1 \leq X_0$. In addition, we have by (3.47)

$$X_1 - X = \sum_{N=0}^{\infty} [(A-BK_1)^T]^N[(K_1-K)^T(R+B^TXB)(K_1-K)](A-BK_1)^N \geq 0.$$

Hence, X_1 is bounded above and below and, therefore, has finite norm. Thus, $(A-BK_1)$ is stable so X_1 satisfies (3.43) with k=1. Repeating the above argument for k=2,3,... yields the desired result.

2) Taking the limit of (3.43) as $k\to\infty$, we obtain

$$X_\infty = A^TX_\infty A - A^TX_\infty B(R+B^TX_\infty B)^{-1}B^TX_\infty A + C^TC. \tag{3.50}$$

Since X is the unique non-negative definite solution of (3.50), $X_\infty=X$.

3) Set $K_1 = (R+B^TX_kB)^{-1}B^TX_kA$ and $K_2 = (R+B^TXB)^{-1}B^TXA$ in (3.47), and take the norm to find

$$\|X_{k+1}- X\| \leq \|X_k- X\|^2\|B(R+B^TXB)^{-1}B^T\|\sum_{N=0}^{\infty}\|(A-BK_{k+1})^{N+1}\|^2.$$

Since $(A-BK_{k+1})$ is stable, it can be shown that, uniformly in k,

$$\sum_{N=0}^{\infty}\|(A-BK_{k+1})^{N+1}\|^2 \leq \text{constant} = C_1$$

Let $C_2 = C_1\|B(R+B^TXB)^{-1}B^T\|$, and the proof is complete.

It should be noted that another iterative scheme is possible here, as in the continuous case, where one solves for a correction term to the solution at each iteration. This procedure faces the same computational drawbacks as in the continuous case and, therefore, will not be discussed further.

Before we extend Theorem 3.4 to the general case, we first want to show that solving the GARE (2.48), which results from the generalized state space system (2.40) with criterion (2.41) and then applying the optimal feedback (2.49), is equivalent to solving a "standard" regulator problem when E is invertible.

Theorem 3.5: Solving the generalized regulator problem of section 2.3.3 is equivalent to solving the following "standard" problem

$$x_{k+1} = \bar{A}x_k + \bar{B}u_k$$

$$y_k = Cx_k \tag{3.51}$$

with criterion

$$J = \frac{1}{2}\sum_{k=0}^{\infty}(y_k^T Q y_k + 2x_k^T S u_k + u_k^T R u_k) \tag{3.52}$$

where

$$\bar{A} = E^{-1}A$$

$$\bar{B} = E^{-1}B \quad . \tag{3.53}$$

Proof: The ARE corresponding to this problem is

$$\begin{aligned}\bar{X} &= \bar{A}^T\bar{X}\bar{A} - (\bar{A}^T\bar{X}\bar{B}+\bar{S})(R+\bar{B}^T\bar{X}\bar{B})^{-1}(\bar{A}^T\bar{X}\bar{B}+S)^T + C^TQC \\ &= A^TE^{-T}\bar{X}E^{-1}A - (AE^{-T}\bar{X}E^{-1}B+S)(R+B^TE^{-T}\bar{X}E^{-1}B)^{-1}(A^TE^{-T}\bar{X}E^{-1}B+S)^T \\ &\quad + C^TQC \quad .\end{aligned} \tag{3.54}$$

The optimal control law is

$$\begin{aligned}u_k^0 &= -(R+\bar{B}^T\bar{X}\bar{B})^{-1}(\bar{A}^T\bar{X}\bar{B}+S)^Tx_k \\ &= -(R+B^TE^{-T}\bar{X}E^{-1}B)^{-1}(A^TE^{-T}\bar{X}E^{-1}B+S)^Tx_k\end{aligned} \tag{3.55}$$

which results in the closed-loop system

$$x_{k+1} = E^{-1}[A-B(R+B^TE^{-T}\bar{X}E^{-1}B)^{-1}(A^TE^{-T}\bar{X}E^{-1}B+S)^T]x_k \quad . \tag{3.56}$$

Now, if we note that

$$\bar{X} = E^T\bar{X}E \tag{3.57}$$

then (3.54) is equivalent to (2.48), and (3.56) is equivalent to (2.40) with the feedback (2.49); that is,

$$Ex_{k+1} = [A-B(R+B^TXB)^{-1}(A^TXB+S)^T]x_k. \tag{3.58}$$

Now we will extend Theorem 3.4 to the general case.

Theorem 3.6: Let X_k, $k=0,1,\ldots$ be the unique non-negative definite solution of the linear algebraic equation

$$E^TX_kE = (A-BK_k)^TX_k(A-BK_k) + K_k^TRK_k + C^TQC - SK_k - (SK_k)^T \tag{3.59}$$

where, recursively,

$$K_k = (R+B^TX_{k-1}B)^{-1}(B^TX_{k-1}A+S^T),\ k=1,2,\ldots, \tag{3.60}$$

and K_0 is chosen such that the closed loop system matrix $E^{-1}(A-BK_0)$ is stable. Then

1) $0 \leq X \leq X_{k+1} \leq X_k \leq \ldots \leq X_0$

2) $\lim_{k\to\infty} X_k = X$

3) in the vicinity of X, $\|X_{k+1} - X\| \leq C_2\|X_k - X\|^2$

where X solves the GARE (2.48), and C_2 is a finite constant.

Proof: 1) We employ the results of Theorem 3.5 to convert the problem; from (3.57) let

$$X_k = E^{-T}\bar{X}_kE^{-1} \tag{3.61}$$

and substitute in (3.59) and (3.60) to obtain

$$\bar{X}_k = [E^{-1}(A-B\bar{K}_k)]^T\bar{X}_k E^{-1}(A-B\bar{K}_k) + \bar{K}_k^T R\bar{K}_k + C^T QC$$

$$- S\bar{K}_k - (S\bar{K}_k)^T \tag{3.62}$$

where

$$\bar{K}_k = (R+B^T E^{-T}\bar{X}_{k-1}E^{-1}B)^{-1}(B^T E^{-T}\bar{X}_{k-1}E^{-1}A+S^T) \ . \tag{3.63}$$

Then

$$\bar{X}_k = \sum_{N=0}^{\infty} [(E^{-1}(A-B\bar{K}_k))^T]^N[\bar{K}_k^T R\bar{K}_k + C^T QC - S\bar{K}_k - (S\bar{K}_k)^T][E^{-1}(A-B\bar{K}_k)]^N \tag{3.64}$$

when $E^{-1}(A-B\bar{K}_k)$ is stable. It can be shown that

$$\bar{X}_1 - \bar{X}_2 = \sum_{N=0}^{\infty}[(E^{-1}(A-B\bar{K}_1))^T]^N[(\bar{K}_1-\bar{K}_2)^T(R+B^T E^{-T}\bar{X}_2 E^{-1}B)(\bar{K}_1-\bar{K}_2)$$

$$+ ((R+B^T E^{-T}\bar{X}_2 E^{-1}B)\bar{K}_2 - B^T E^{-T}\bar{X}_2 E^{-1}A - S^T)^T(\bar{K}_1-\bar{K}_2)$$

$$+ (\bar{K}_1-\bar{K}_2)^T((R+B^T E^{-T}\bar{X}_2 E^{-1}B)\bar{K}_2 - B^T E^{-T}\bar{X}_2 E^{-1}A - S^T)]$$

$$[E^{-1}(A-B\bar{K}_1)]^N \tag{3.65}$$

or, alternatively

$$\bar{X}_1 - \bar{X}_2 = \sum_{N=0}^{\infty}[(E^{-1}(A-B\bar{K}_2))^T]^N[(\bar{K}_1-\bar{K}_2)^T(R+B^T E^{-T}\bar{X}_1 E^{-1}B)(\bar{K}_1-\bar{K}_2)$$

$$+ ((R+B^T E^{-T}\bar{X}_1 E^{-1}B)\bar{K}_2 - B^T E^{-T}\bar{X}_1 E^{-1}A - S^T)^T(\bar{K}_1-\bar{K}_2)$$

$$+ (\bar{K}_1-\bar{K}_2)^T((R+B^T E^{-T}\bar{X}_1 E^{-1}B)\bar{K}_2 - B^T E^{-T}\bar{X}_1 E^{-1}A - S^T)]$$

$$[E^{-1}(A-B\bar{K}_2)]^N. \tag{3.66}$$

Now let $\bar{X}_0$ satisfy (3.62) for a $\bar{K}_0$ chosen such that $E^{-1}(A - B\bar{K}_0)$ is stable. Let $\bar{X}_1$ be the associated solution to (3.64). Using (3.66) we obtain

$$\bar{X}_0 - \bar{X}_1 = \sum_{N=0}^{\infty} \left[\left(E^{-1}(A-B\bar{K}_1)\right)^T\right]^N \left[(\bar{K}_0-\bar{K}_1)^T(R+B^TE^{-T}\bar{X}_0E^{-1}B)(\bar{K}_1-\bar{K}_2)\right]$$

$$\left[E^{-1}(A-B\bar{K}_1)\right]^N \geq 0$$

so that $\bar{X}_1 \leq \bar{X}_0$. In addition, we have by (3.65)

$$\bar{X}_1 - \bar{X} = \sum_{N=0}^{\infty} \left[\left(E^{-1}(A-B\bar{K}_1)\right)^T\right]^N \left[(\bar{K}_1-\bar{K})^T(R+B^TE^{-T}\bar{X}E^{-1}B)(\bar{K}_1-\bar{K})\right]$$

$$\left[E^{-1}(A-B\bar{K}_1)\right]^N \geq 0.$$

Hence, $\bar{X}_1$ is bounded above and below and, therefore, has finite norm. Thus, $E^{-1}(A-B\bar{K}_1)$ is stable so $\bar{X}_1$ satisfies (3.62) with k=1. Repeating the above argument for k=2,3,... yields

$$0 \leq \bar{X} \leq \bar{X}_{k+1} \leq \bar{X}_k \leq \ldots \leq \bar{X}_0 .$$

Since by (3.61) E is a congruence transformation on X to yield $\bar{X}$, the desired result is implied.

2) Taking the limit of (3.59) as $k \to \infty$ we obtain

$$E^TX_\infty E = A^TX_\infty A - (A^TX_\infty B+S)(R+B^TX_\infty B)^{-1}(A^TX_\infty B+S)^T + C^TQC . \quad (3.67)$$

Since X is the unique non-negative definite solution to (3.67), $X_\infty = X$.

3) It is easily shown from (3.65) that

$$E^T(X_1-X_2)E = \sum_{N=0}^{\infty} [(A-BK_1)^T]^N[(K_1-K_2)^T(R+B^TX_2B)(K_1-K_2)$$

$$+ \left((R+B^TX_2B)K_2 - B^TX_2A-S^T\right)^T(K_1-K_2)$$

$$+ (K_1-K_2)^T\left((R+B^TX_2B)K_2 - B^TX_2A-S^T\right)][A-BK_1]^N. \tag{3.68}$$

Set $K_1 = (R+B^TX_kB)^{-1}(B^TX_kA+S^T)$ and $K_2 = (R+B^TXB)^{-1}(B^TXA+S^T)$

in (3.68), and take norms to find

$$\|X_{k+1} - X\| \leq \|X_k-X\|^2 \|E^{-1}\|^2\|B(R+B^TXB)^{-1}B^T\| \sum_{N=0}^{\infty}\|(A-BK_{k+1})^{N+1}\|^2.$$

Since $(A-BK_{k+1})$ is stable, it can be shown that, uniformly in k,

$$\sum_{N=0}^{\infty} \|(A-BK_{k+1})^{N+1}\|^2 \leq \text{constant} = C_1.$$

Let $C_2 = C_1 \|E^{-1}\|^2\|B(R+B^TXB)^{-1}B^T\|$, and the proof is complete.

CHAPTER 4

CONDITIONING OF THE ALGEBRAIC RICCATI PROBLEM

As stated in Chapter 2, it is desirable to associate a measure with a computing problem which reflects the overall sensitivity of the solution to changes in the data. It is the intent of this chapter to derive such a measure (i.e., condition number) for the problem of computing a solution to the GARE. To be most useful, the condition number must be easily computable. It should not require additional computations on the same order of computing the solution itself.

We will first state the desired form for the condition number and review previous work on condition estimates for the Riccati problem. Then first order perturbation analysis will be employed to derive new condition estimates for the continuous- and discrete-time Riccati equations. Ways of employing balancing to improve the numerical accuracy of the calculated solution will be discussed. An efficient method for incorporating a change of model coordinates is presented.

4.1 Previous Work

We seek a relative condition estimate for the Riccati problem in the following form:

$$\frac{\|X - \bar{X}\|}{\|X\|} = C \cdot \kappa(X) \cdot \varepsilon_m \ , \tag{4.1}$$

where X is the exact solution to the GARE, $\bar{X}$ is the computed solution, C is a constant possibly depending on the size of the problem, $\kappa(X)$ is the desired condition number, and ε_m is the machine epsilon (precision). Machine epsilon is defined as the smallest positive

number that can be added to 1.0 such that the machine recognizes that the sum is different from 1.0. In this form, if $\kappa(X)$ is $0(10^N)$, then the computed solution can be expected to differ from the true solution by N significant digits.

We should note here that this type of indicator of solution accuracy is desired because the more traditional method of examining the size of the residual is not always reliable. The residual is the remainder quantity obtained when the computed solution is substituted in the original problem. That is, for the GARE (2.30) the residual is

$$\text{residual} := (A-BR^{-1}S^T)^T\bar{X}E + E^TX(A-BR^{-1}S^T) - E^T\bar{X}BR^{-1}B^T\bar{X}E + C^TQC - SR^{-1}S^T . \quad (4.2)$$

Stewart [15] explores in detail the behavior of the residual in the linear equations problem $Ax=b$ and cites examples where the residual $:= b - A\bar{x}$ is not a reliable indicator of solution ($\bar{x}$) accuracy. There is no reason to expect that the more complex problem of the GARE produces residuals that are better behaved, even though counter-examples are more difficult to construct.

We look for $\kappa(X)$ of the form:

$$\kappa(X) = f(A,B,C,E,Q,R,S) . \quad (4.3)$$

That is, $\kappa(X)$ should be a function of the matrices involved in the GARE. We want the norms involved in (4.1) or in the computation of $\kappa(X)$ to be the 1, ∞ or Frobenius norm versus the 2 norm to minimize the number of calculations involved.

There have been several attempts to derive bounds for the solution to the ARE [29]-[34], none of which are of the form (4.1). The following result by Byers [20] is for the continuous-time ARE (2.27):

$$\kappa_B(X) = \frac{\|C^T Q C\| + 2\|A\|\|X\| + \|BR^{-1}B^T\|\|X\|^2}{\|X\| SEP[A_c^T, -A_c]} \tag{4.4}$$

where

$$A_c = A - BR^{-1}B^T X \tag{4.5}$$

$$SEP(F,G) := \inf_{\|P\|=1} \|PF - GP\| \ . \tag{4.6}$$

Byers arrives at this bound via a first-order-perturbation approximation to (2.27). A notable feature of this condition estimate is that it includes the effect of the separation of the closed-loop spectrum (4.6) on the condition of the problem. For a detailed definition of SEP(F,G) and a discussion of its properties see Stewart [35],[36]. Some speculation [6], [20] and empirical results suggest that a small separation of the closed-loop spectrum causes loss of numerical accuracy in the computed solution (see Chapter 5). A potential drawback to Byers' result (4.4) is the appearance of $\|X\|^2$ in the numerator and $\|X\|$ in the denominator. This would suggest ill-conditioning of the problem when $\|X\|$ is large or small, which is not necessarily true.

Martensson [5] showed that V_{11}, which must be inverted to form the Riccati (2.27) solution, is singular if the model is unstabilizable for the case when eigenvectors are used as the basis for the stable eigenspace in the solution process. Laub [7], Pappas, et.al. [8], and Emami-Naeini [9] prove analogous results when Schur vectors are used as

the basis for the stable eigenspace. This fact and the fact that V_{11} must be inverted (that is, a linear system of the form $XV_{11} = V_{21}$ must be solved for X) has prompted speculation [6], [7] that the condition of V_{11} with respect to inversion, $\kappa(V_{11})$, might be a good indicator of the condition of the Riccati problem (2.27). Indeed, empirical results show that when V_{11} is ill-conditioned with respect to inversion, say $\kappa(V_{11})$ is $O(10^N)$; then, in general, N digits of accuracy are lost in the solution X. Since for the GARE solution, inversion of V_{11} involves the inversion of E; then $\kappa(V_{11})$ can be expected to reflect any ill-effects that near singularity of E might have on the solution. However, examples do exist (see Chapter 5) where the Riccati problem is ill-conditioned, but this fact is not indicated by the magnitude of $\kappa(V_{11})$.

Another indicator of numerical accuracy in the solution of the ARE has been suggested by Paige and Van Loan [37]. Their reasoning is based on the following result:

Theorem 4.1: Let the unitary matrix $E \in C^{nxn}$ be partitioned in the form

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}$$

where $W_{11} \in C^{mxm}$ with $m \leq \frac{n}{2}$. Then there are unitary matrices $U = \text{diag}(U_1, U_2)$ and $V = \text{diag}(V_1, V_2)$; $U_1, V_1 \in C^{mxm}$; $U_2, V_2 \in C^{(n-m)x(n-m)}$ such that

$$U^H W V = \begin{bmatrix} \Gamma & -\Delta & 0 \\ \Delta & \Gamma & 0 \\ 0 & 0 & I \end{bmatrix} , \qquad (4.7)$$

where

$$\Gamma = \text{diag}\ (\gamma_1,\gamma_2,\ldots,\gamma_m) \geq 0$$

$$\Delta = \text{diag}\ (\delta_1,\delta_2,\ldots,\delta_m) \geq 0$$

$$\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_m \leq 1 \quad . \tag{4.8}$$

Proof: [38]

The use of this theorem in [37] suggests that if we perform this special singular value decomposition on the orthogonal Z found in our reduction of the generalized eigenproblem (2.59) we have

Theorem 4.2: Given Z_{11}, Z_{21} as found in (2.59), (2.61)

1) Then there exists orthogonal U and V ε R^{nxn} such that

$$U^T Z_{11} V = \Gamma = \text{diag}\ (\gamma_1,\gamma_2,\ldots,\gamma_n)\ ;$$

$$U^T Z_{21} V = \Delta = \text{diag}\ (\delta_1,\delta_2,\ldots,\delta_n)\ ;$$

$$\gamma_i^2 + \delta_i^2 = 1;\quad 0 \leq \gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_n \leq 1\ . \tag{4.9}$$

2) If Γ is nonsingular, $X = U\ \text{diag}(\frac{\delta_1}{\gamma_1},\ldots,\frac{\delta_n}{\gamma_n})\ U^T$ solves the ARE (2.27).

Proof: [37]

The size of γ_1 is suggested in [37] as being an indicator of potential numerical difficulties in the solution of the ARE because

$$\|X\|_2 = \left|\frac{\delta_1}{\gamma_1}\right| = (1-\gamma_1^2)^{1/2}/\gamma_1 = \text{cotan}\ \gamma_1 \tag{4.10}$$

and if X is computed on a computer having a machine precision ε, then rounding errors of order ε/γ_1 can be expected to contaminate the result. Therefore, $1/\gamma_1$ may indicate the condition of the ARE.

We can relate $1/\gamma_1$ to $\kappa(Z_{11})$ ($\kappa(Z_{11}) = \kappa(V_{11})$ for the ARE (2.27)), which was suggested previously as a condition measure. Recall,

$$\kappa(Z_{11}) = \|Z_{11}\|_2 \, \|Z_{11}^{-1}\|_2$$

$$= \max(\sigma(Z_{11})) \, \frac{1}{\min(\sigma(Z_{11}))} \tag{4.11}$$

where $\sigma(Z_{11})$ denotes the set of singular values of Z_{11} [15]. Therefore,

$$\kappa(Z_{11}) = \frac{\gamma_n}{\gamma_1} \leq \frac{1}{\gamma_1} \quad . \tag{4.12}$$

So we see the two estimates are of the same order unless all of the singular values of Z_{11} are small.

4.2 First-Order-Perturbation Analysis of the GARE

To facilitate our analysis of the continuous-time Riccati problem in this section, we start with the GARE (2.30) and make the following definitions:

$$D := A - BR^{-1}S^T$$

$$G := BR^{-1}B^T$$

$$H := C^TQC - SR^{-1}S^T \quad . \tag{4.12}$$

The GARE can now be written as

$$0 = D^TXE + E^TXD - E^TXGXE + H =: F_c(X) \quad . \tag{4.13}$$

Suppose that there is a small perturbation in $F_c(X)$, say δF_c; we would like to know the effect of this perturbation on the solution X.

Theorem 4.3: For consistent matrix norms, and given a small perturbation δF_c to (4.13), we have for the resultant change in the solution δX

$$\|\delta X\| \geq \frac{\|\delta F_c\|}{\|E^T\|\|A_c\|+\|E\|\|A_c^T\|} \tag{4.14}$$

and

$$\|\delta X\| \leq \frac{\kappa(E)\kappa(E^T)\|\delta F_c\|}{\|E\|\|E^T\|SEP[A_c E^{-1},-(A_c E^{-1})^T]} \tag{4.15}$$

where

$\kappa(E) := \|E\|\|E^{-1}\|$ = condition of E with respect to inversion

$A_c := D - GXE$ = closed-loop-system matrix.

Proof: Starting with (4.13), define

$$F_c(X-\delta X) := F_c(X) + \delta F_c \tag{4.16}$$

$$= D^T(X-\delta X)E + E^T(X-\delta X)D - E^T(X-\delta X)G(X-\delta X)E + H.$$

Neglecting second-order terms in δX,

$$F_c(X-\delta X) = D^T XE + E^T XD - E^T XGXE + H - (D-GXE)^T\delta XE$$

$$- E^T\delta X(D-GXE)$$

$$= F_c(X) - (D-GXE)^T\delta XE - E^T\delta X(D-GXE) \quad .$$

Therefore,

$$-\delta F_c = A_c^T\delta XE + E^T\delta XA_c \tag{4.17}$$

Taking norms in (4.17), we have

$$\|\delta F_c\| \leq \|A^T_c\|\|\delta X\|\|E\| + \|E^T\|\|\delta X\|\|A_c\|$$

which yields (4.14). We pre-multiply (4.17) by E^{-T} and post-multiply by E^{-1} to obtain

$$-E^{-T}\delta F_c E^{-1} = (A_c E^{-1})^T \delta X + \delta X A_c E^{-1} .$$

Taking norms we have

$$\|E^{-T}\delta F_c E^{-1}\| = \frac{\|ZA_c E^{-1}+(A_c E^{-1})^T Z\|}{\|Z\|} \|\delta X\|$$

where $Z = \frac{\delta X}{\|\delta X\|}$. Therefore,

$$\|E^{-T}\|\|E^{-1}\|\|\delta F_c\| \geq \inf_{\|P\|=1} \frac{\|PA_c E^{-1}+(A_c E^{-1})^T P\|}{\|P\|} \|\delta X\|$$

$$= \|\delta X\| SEP[A_c E^{-1}, -(A_c E^{-1})^T]$$

which yields (4.15).

Theorem 4.4: Given the GARE (2.30), a relative condition number for the solution X is

$$\kappa A_c(X) := \frac{\kappa(E)\kappa(E^T)\|C^T QC - SR^{-1}S^T\|}{\|X\|\|E\|\|E^T\| SEP[A_c E^{-1}, -(A_c E^{-1})^T]} \tag{4.18}$$

where $A_c = (A - BR^{-1}S^T - BR^{-1}B^T XE)$.

Proof: To facilitate the analysis, we consider perturbations in $F_c(X)$ on the order of a perturbation in the constant term, i.e.,

$$\|\delta F_c\| \leq \|H\| \cdot \varepsilon_m , \tag{4.19}$$

so that from Theorem 4.3 we obtain

$$\frac{\|\delta X\|}{\|X\|} \leq \frac{\kappa(E)\kappa(E^T)\|H\|}{\|X\|\|E\|\|E^T\| SEP[A_c E^{-1}, -(A_c E^{-1})^T]} \cdot \varepsilon_m . \tag{4.20}$$

From (4.20) we can make the following definition:

$$\kappa A_c(X) := \frac{\kappa(E)\kappa(E^T)\|H\|}{\|X\|\|E\|\|E^T\| \ SEP[A_c E^{-1}, -(A_c E^{-1})^T]} \tag{4.21}$$

:= relative condition number for the continuous-time GARE.

We remark here that the condition number of Theorem 4.4 has the advantage that it includes the effects of ill-conditioning of E with respect to inversion, the separation of the closed-loop spectrum and the ill-conditioning of R with respect to inversion when S is non-zero. It does have the disadvantage of having $\|X\|$ in the denominator, as did Byers' result. Clearly, the form of the bound on the perturbation, δF_c, dictates the form of the resulting condition number. One could consider perturbations to all matrices involved in the problem with a corresponding increase in the complexity of the resulting condition number. However, at this point it is not clear that this is necessary, and it is counter productive to our goal of a simple, easily computable measure. The proper form of the bound on δF_c is a topic of continuing research.

We now turn our attention to the discrete-time Riccati problem. To facilitate our analysis, we start with the GARE (2.48) and make the following definition:

$$F_d(X) := A^T XA - E^T XE - (A^T XB+S)(R+B^T XB)^{-1}(A^T XB+S)^T + C^T QC = 0 . \tag{4.22}$$

As in the continuous-time case, suppose there is a small perturbation in $F_d(X)$, say δF_d, we would like to know the effect of this perturbation on the solution X.

Theorem 4.5: For consistent matrix norms, and given a small perturbation δF_d to (4.22), we have for the resultant change in the solution δX

$$\|\delta X\| \geq \frac{\|\delta F_d\|}{\|E^T\|\|E\| + \|A_d^T\|\|A_d\|} \tag{4.23}$$

and

$$\|\delta X\| \leq \frac{\|\delta F_d\|}{\frac{\|E\|\|E^T\|}{\kappa(E)\kappa(E^T)} - \|A_d^T\|\|A_d\|} \tag{4.24}$$

if

$$\|E\|\|E^T\| \geq \kappa(E)\kappa(E^T)\|A_d^T\|\|A_d\| \tag{4.25}$$

where

$\kappa(E) := \|E\|\|E^{-1}\|$ = condition of E with respect to inversion

$A_d := (A-BK)$ = closed-loop-system matrix

$K := (R+B^TXB)^{-1}(A^TXB+S)^T$.

Proof: Starting with (4.22) define

$$F_d(X+\delta X) := F_d(X) + \delta F_d$$

$$= A^T(X+\delta X)A - E^T(X+\delta X)E + C^TQC$$

$$- [A^T(X+\delta X)B+S][R+B^T(X+\delta X)B]^{-1}[A^T(X+\delta X)B+S]^T$$

After expanding, neglecting second- or higher-order terms in δX, and some algebraic manipulation we obtain

$$F_d(X+\delta X) = F_d(X) - E^T\delta XE + (A-BK)^T\delta X(A-BK) \ .$$

Therefore,

$$\delta F_d = -E^T\delta XE + A_d{}^T\delta XA_d \ . \tag{4.26}$$

Taking norms in (4.26) we have

$$\|\delta F_d\| \leq \|E^T\|\|\delta X\|\|E\| + \|A_d{}^T\|\|\delta X\|\|A_d\|$$

which yields (4.23). We pre-multiply (4.26) by E^{-T}, and post-multiply by E^{-1} to obtain

$$E^{-T}\delta F_dE^{-1} = -\delta X + (A_dE^{-1})^T\delta XA_dE^{-1} .$$

or

$$\delta X = (A_dE^{-1})^T\delta XA_dE^{-1} - E^{-T}\delta F_dE^{-1} \ .$$

Taking norms yields (4.24).

Theorem 4.6: Given the GARE (2.48), a relative condition number for the solution X is

$$\kappa A_d(X) := \frac{\dfrac{\|C^TQC\|}{\|X\|}}{\dfrac{\|E\|\|E^T\|}{\kappa(E)\kappa(E^T)} - \|A_d{}^T\|\|A_d\|} \tag{4.27}$$

where $A_d = A - B(R+B^TXB)^{-1}(A^TXB+S)^T$.

Proof: With reasoning similar to that for the continuous-time problem, we consider a perturbation in $F_d(x)$ on the order of a perturbation in the constant term, i.e.,

$$\|\delta F_d\| \leq \|C^TQC\| \cdot \epsilon_m \tag{4.28}$$

and the desired result follows from Theorem 4.5.

We remark here that when E=I, the requirement (4.25) becomes $1 > \|A_d^T\|\|A_d\|$. This is always true for "some" norm since A_d is the closed-loop-system matrix and all of its eigenvalues have magnitutde less than unity. Therefore, its spectral radius is less than one and there always exists a norm that is arbitrarily close to the spectral radius $\rho(A_d)$ ([15], Theorem 6.3.8). Thus, one could argue in this case that

$$\kappa A_d(X) \approx \frac{\|C^T Q C\|}{\|X\|(1-\rho^2(A_d))}$$

and the term $1-\rho^2(A_d)$ would have the effect of raising the condition number when the spectrum of the closed-loop system has a pole near the unit circle. For the case $E \neq I$, (4.25) can be a very restrictive requirement and is a subject of continuing research.

4.3 Balancing to Improve Condition

Considerable success has been attained in increasing the numerical accuracy of the solutions to linear equations and eigenvalue problems by appropriate scaling of problem parameters [39], [40]. Therefore, it is reasonable to expect that some sort of balancing or problem scaling could improve the accuracy of the numerical computations involved in computing the solution to the GARE and, hence, the overall "condition" of the problem. One could argue that since the numerical solution depends fundamentally on the QZ algorithm for the generalized eigenproblem [41], that balancing the eigensystem for the generalized eigenproblem $\lambda Ly = My$ could improve the Riccati solution. Ward [42]

has proposed a balancing algorithm specifically designed to precede QZ-type algorithms. This balancing consists of permutations and two sided diagonal transformations. The strategy is to scale L and M so that their elements have magnitudes as close to unity as possible. Thus, the large elements of L and M cannot mask the effect of the small elements, as can often be the case.

Those interested in the details of the algorithm are referred to [42] where the scaling strategy is discussed. Numerical results for example eigenproblems are also given in [42]. Numerical experience in solving the GARE has shown that this balancing strategy can increase the accuracy of the solution; an example is given in Chapter 5. This balancing can also increase the reliability of $\kappa(V_{11})$ as an indicator of condition of the Riccati problem. However, one does pay the price in increased computational time, about 10% of the QZ algorithm computation time.

An alternate approach to the direct balancing of the eigenproblem is to attempt some sort of coordinate change in the problem which generates the Riccati equation. That is, some nonsingular transformation T to change coordinates in the original model as follows:

$$x(t) = Tw(t) \tag{4.29}$$

or

$$x_k = Tw_k \ . \tag{4.30}$$

There is reason to believe that the coordinate balancing transformation proposed by Moore [43] may result in a transformed Riccati equation that is more amenable to numerical solution. Moore has shown [44] that

by a change of coordinates, one can scale the relative size of components of $e^{At}B$ and $e^{A^T t}C^T$. Since, as we have already pointed out, stabilizability of the model influences the computational accuracy of the Riccati solution, one could argue that a change of coordinates increasing the relative size of components of $e^{At}B$ would be beneficial prior to computing the Riccati solution. Specifically, Moore's "internally balanced" coordinates in which the reachability and observability gramians are equal and diagonal or "input-normal" coordinates where the reachability gramian is identity are logical choices.

Laub [6] illustrated the effect of this coordinate type balancing on the following linear optimal control problem: Find a feedback controller $u(t)=Kx(t)$ which minimizes the performance index

$$J(u) = \int_0^\infty [x^T(t)Qx(t) + u^T(t)Ru(t)]dt$$

with plant dynamics given by

$$\dot{x}(t) = Ax(t) + Bu(t); \quad x(0) = x_0 .$$

Assume $Q=Q^T \geq 0$, $R=R^T>0$, (A,B) stabilizable and (A,C) detectable, where $C^TC=Q$ and rank (C) = rank (Q). Then the optimal control is well known to be

$$u(t) = -R^{-1}B^TXx$$

where X solves the ARE

$$A^TX + XA - XBR^{-1}B^TX + Q = 0 .$$

Suppose we change coordinates via (4.29). Then in terms of the new state w(t), our problem is to minimize

$$\int_0^\infty [w^T(t)T^TQTw(t) + u^T(t)Ru(t)]dt \tag{4.31}$$

subject to

$$\dot{w}(t) = (T^{-1}FT)w(t) + (T^{-1}B)u(t). \tag{4.32}$$

The associated solution X_w of the transformed Riccati equation is related to the original X by

$$X = T^{-T}X_wT^{-1} \tag{4.33}$$

One can see from (4.32) and (4.33) that if T is ill-conditioned with respect to inversion and the balancing is applied as these equations indicate, then this technique can potentially introduce more error into the solution than would originally have appeared. Hence, the opposite of the intended effect could occur. Of course, one could be careful to only choose a well-conditioned T, like a diagonal scaling matrix for instance. However, if T is a modal or system balancing [43] transformation, it could be quite ill-conditioned.

We can reduce this problem significantly in our generalized problem formulation framework as the following result will show.

Theorem 4.7: If a change of coordinates of the form (4.29) is made in the continuous-time generalized optimal regulator problem of section 2.3.1 or a change of coordinates of the form (4.30) is made in the discrete-time generalized optimal regulator problem of Section 2.3.3, then the solution X to the original problem may be found as follows:

1) replace A by AT, C by CT, S by $T^T S$ and E by ET in the appropriate matrix pencils ((2.51) or (2.65) for the continuous problem, (2.67) or (2.68) for the discrete problem)

2) determine the appropriate $\bar{P}$, $\bar{Z}$ transformations for the modified pencils and partition $\bar{Z}$ as in (2.61)

3) let $\bar{V} = \begin{bmatrix} ET & 0 \\ 0 & I \end{bmatrix} \bar{Z}$, then $X = \bar{V}_{21}\bar{V}_{11}^{-1}$ (4.34)

solves the original GARE ((2.30) for the continuous problem, (2.48) for the discrete problem) with $X = X^T \geq 0$.

Proof: For the continuous-time case, the problem becomes in the transformed coordinates

$$\text{System:} \quad ET\,\dot{w}(t) = ATw(t) + Bu(t)$$
$$y(t) = CT\,w(t) \tag{4.35}$$

$$\text{Criterion:} \quad J = \frac{1}{2}\int_0^\infty (y^T Qy + u^T Ru + 2w^T T^T Su)dt. \tag{4.36}$$

Application of Hamilton-Jacobi theory as in Section 2.3.1 gives rise to a set of equations to which we can associate the following matrix pencil:

$$\lambda \begin{bmatrix} ET & 0 \\ 0 & T^T E^T \end{bmatrix} - \begin{bmatrix} (A-BR^{-1}S^T)T & -BR^{-1}B^T \\ -T^T(C^TQC-SR^{-1}S^T)T & -T^T(A-BR^{-1}S^T)^T \end{bmatrix}. \tag{4.37}$$

This pencil can also be obtained by performing step 1 of the theorem on the pencil associated with (2.57). Now we can factor (4.37) as

$$\begin{bmatrix} I & 0 \\ 0 & T^T \end{bmatrix} \left[\lambda \begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix} - \begin{bmatrix} (A-BR^{-1}S^T) & -BR^{-1}B^T \\ -(C^TQC-SR^{-1}S^T) & -(A-BR^{-1}S^T)^T \end{bmatrix} \right] \begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix} \tag{4.38}$$

If we perform the ordered transformation on (4.38) by finding a $\bar{P}$ and $\bar{Z}$ as in (2.60) (as stated in step 2 of the theorem) and compare the result with (2.60), we find

$$Z = \begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix} \bar{Z} \ . \tag{4.39}$$

Recall from Theorem 2.4 that

$$X = V_{21}V_{11}^{-1} = Z_{21}(EZ_{11})^{-1} \ . \tag{4.40}$$

Substitute (4.39) into (4.40) and the theorem is proved for the continuous case. An analogous procedure proves the discrete case. We omit those details.

Note from the theorem that the numerous occurrences of the inverse of T have been eliminated. The inversion of T is essentially required only once in the process of solving the linear system

$$X\bar{V}_{11} = \bar{V}_{21}$$

for the Riccati solution X. Computational advantage has also been realized because the number of matrix multiplications required in the solution process has been reduced.

CHAPTER 5

NUMERICAL EXPERIMENTS

The methods for solution of the GARE presented in Chapter 2, the iterative refinement procedure derived in Chapter 3, and the condition estimates for the computed solution of Chapter 4 are all geared to the efficient and reliable numerical computation of the Riccati solution. This chapter describes a FORTRAN software package (RICPACK) developed to aid in the study of the numerical conditioning of the algebraic Riccati equation and other closely related topics. First, a brief description of the package will be given. A short discussion of the algorithms and software will follow. Finally, numerical examples and results will be given to illustrate relevant aspects of the numerical solution.

5.1 Software Package RICPACK

RICPACK was developed to aid in the study of the numerical conditioning of the algebraic Riccati equation, methods for improving the condition, and iterative refinement of the solution. The FORTRAN subroutines in the package were written in modular form and are designed to facilitate their incorporation in some larger computer-aided control system design (CACSD) package or computer-aided systems and control analysis and design environment (CASCADE).

A FORTRAN driver program has also been written, primarily as a research tool, that is user-friendly for use in an interactive "terminal" type environment. The program prompts for all necessary input.

Convenient input default options exist not only for ease of data input, but also for exploitation by the subroutines to reduce the number and complexity of the computations. These options are also designed to speed the input of more "standard" type problems. Table 5.1 lists the default options available.

TABLE 5.1

Default Options

Input Matrix	Default Value
E	Identity
Q	Identity
S	Zero
R	Identity (or enter diagonal elements only, if diagonal)

Highlights of RICPACK capabilities include:

(a) Choice of calculation of the stabilizing (non-negative definite), anti-stabilizing (non-positive definite), or just any (possibly) indefinite or nonsymmetric solution to the GARE.

(b) Coordinate or system balancing of the system model [43], [45]; i.e., a special coordinate transformation on (2.18) or (2.40) of the form

$$x(t) = T\, w(t) \tag{5.1}$$

or

$$x_k = T\, w_k \tag{5.2}$$

respectively, such that the observability and reachability gramians of the transformed system are equal and diagonal.

(c) Ward's balancing [42] of the generalized eigenproblem (2.58) or (2.68) (prior to the QZ transformation) which consists of permutations and two-sided-diagonal transformations.

(d) Direct handling of singular control weighting or singular measurement noise covariance by compression of the extended pencils (2.65) or (2.67).

(e) Direct handling of cross-weighting or noise correlation; i.e., $S \neq 0$.

(f) Provision for robustness recovery procedure; i.e., to replace the C^TQC term in the GARE with $Q + \gamma C^TC$ and iterate on γ, the driver program need only modify one block of the matrix pencil at each iteration.

(g) Iterative refinement (or new solutions for small parameter perturbations) by Newton's method and Sylvester equations; i.e., iteratively solving equations (3.27) or (3.59). As of this writing, this is implemented for E = identity only because although algorithms exist for solving these general Sylvester equations, reliable software implementing these algorithms is not readily available.

(h) Model unstabilizability detection as indicated by the condition of V_{11} with respect to inversion in (2.62) or (2.69); V_{11} is singular for an unstabilizable model.

(i) Calculation of unique stabilizing solution for stabilizable models with undetectable modes.

(j) Residual calculation of the form

$$r = \frac{\|\text{Residual}\|_1}{\|X\|_1}. \tag{5.3}$$

(k) Condition estimates for the Riccati problem derived in Chapter 4.

5.2 Algorithms and Software

Most of the algorithmic computations (i.e., those beyond matrix algebra) performed in RICPACK employ proven stable algorithms coded into reliable, portable FORTRAN software. Two sources for the software are LINPACK [39] and EISPACK [40], [46]. Modified LINPACK software is used for linear equation solving (the BLAS are replaced with in-line code), estimating the condition of a non-diagonal cost matrix R, solving the system $XV_{11} = V_{21}$ and in estimating the condition of V_{11}. The LINPACK singular value decomposition (SVD) is used in compressing the $(2n+m) \times (2n+m)$ pencil when necessary. EISPACK-type QZ software is used for performing the QZ transformation and generalized eigenvalue calculations. Other EISPACK software is used in calculating the coordinate balancing [43], [45] transformation, in addition to software based on the Bartels-Stewart algorithm [47] for Lyapunov equations. The Bartels-Stewart based software is also used in condition estimation and Newton's iteration calculations. Ward's [42] software is used for the eigenproblem balancing, and Van Dooren's software [44] is used for ordering the transformed pencil. A description of each subroutine used is given in the Appendix.

5.3 Numerical Examples

The following simple continuous-time example was used to illustrate the numerical properties of RICPACK when stabilizability is the key factor:

Example 1

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} \varepsilon \\ 0 \end{bmatrix} u$$

$$y = [1 \quad 1]x \tag{5.4}$$

$$\text{minimize } \int_0^\infty (y^T y + u^T u)dt \tag{5.5}$$

This system is stabilizable for $\varepsilon \neq 0$ and completely reconstructible. The applicable ARE is

$$A^T X + XA - XBB^T X + C^T C = 0 \ . \tag{5.6}$$

The "true" solution for X can be hand calculated for comparison purposes as

$$X = \begin{bmatrix} \dfrac{1+\sqrt{1+\varepsilon^2}}{\varepsilon^2} & \dfrac{1}{2+\sqrt{1+\varepsilon^2}} \\ \dfrac{1}{2+\sqrt{1+\varepsilon^2}} & \dfrac{1}{4} - \dfrac{\varepsilon^2}{4(2+\sqrt{1+\varepsilon^2})^2} \end{bmatrix} > 0 \ . \tag{5.7}$$

Note that as $\varepsilon \to 0$ the system approaches unstabilizability and the (1,1) element of X tends to infinity.

The solution to this problem was numerically computed on a DEC KL-10 (under TOPS-20) in double precision using RICPACK. The machine precision is near 10^{-18} in this case. The results of interest are summarized in Table 5.2. The only measure of condition included in the table is the condition of V_{11} with respect to inversion because the other measures did not give an indication that the solution accuracy was degenerating as $\varepsilon \to 0$. We note here that the data in Table 5.2 and the succeeding tables that are expressed as a power of 10 are rounded to the nearest power of 10.

TABLE 5.2

Numerical Results for Example 1, $\epsilon = 10^{-N}$

N	$\kappa(V_{11})$	r (5.3)	Acc*	Newton iterations	r (5.3)	Acc*
0	10^{0}	10^{-18}	17	-	-	-
2	10^{4}	10^{-14}	14	-	-	-
4	10^{8}	10^{-10}	10	2	10^{-18}	17
6	10^{12}	10^{-8}	6	3	10^{-20}	17
8	10^{16}	10^{-2}	2	4	10^{-34}	17
9	10^{18}	10^{-1}	0	6	10^{-18}	17
10	10^{20}	10^{0}	0	-	-	-
The following data includes Ward balancing effects						
0	10^{0}	10^{-18}	17	-	-	-
5	10^{7}	10^{-15}	15	-	-	-
10	10^{12}	10^{-9}	9	2	10^{-18}	17
11	10^{16}	10^{-7}	7	3	0	17
12	10^{17}	10^{-7}	7	3	0	17
13	10^{17}	10^{-6}	7	3	10^{-18}	17
14	sing.	10^{1}	0	-	-	-

*Accuracy in correct significant digits.

Some useful observations can be made on this data. One can see that for this example, $\kappa(V_{11})$ and the residual (r) are both good indicators of the numerical accuracy. Since machine precision is near 10^{-18}, one would expect about 17 correct digits for a well-conditioned problem (which is the case for ϵ=1). The data indicates that one digit

of accuracy is lost for each power of 10 change in $\kappa(V_{11})$ and r. This is desirable behavior of a condition estimate. Note that Ward balancing improves the condition of V_{11} and reduces the value of r for the same value of ϵ. Ward balancing enables solution calculation for smaller values of ϵ, but the accuracy is not as smooth a function of $\kappa(V_{11})$. However, the residual is still a good indicator of accuracy. Note that in all cases with a reasonable starting guess a few iterations of Newton's method restores full accuracy. The generalized eigenvalue solution was used as a starting guess and was considered reasonable if $\kappa(V_{11}) < 1./(\text{machine precision})$. When this condition was not satisfied, the Newton iteration failed to converge to the desired solution.

The above example illustrates that stabilizability of the model does indeed influence the numerical accuracy of the Riccati solution. Also, $\kappa(V_{11})$ and r are good indicators of solution accuracy as the model approaches unstabilizability. However, $\kappa(V_{11})$ may not be a good indicator in other situations as the following example will show:

Example 2

$$\dot{x} = \begin{bmatrix} -\epsilon & 1 & 0 & 0 \\ -1 & -\epsilon & 0 & 0 \\ 0 & 0 & \epsilon & 1 \\ 0 & 0 & -1 & \epsilon \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = [1\ 1\ 1\ 1]x \tag{5.8}$$

$$\text{minimize } \int_0^\infty (y^T y + u^T u)dt \tag{5.9}$$

The model is completely controllable and observable. The open-loop poles are at $\pm\epsilon \pm j$, and the applicable ARE is

$$A^TX + XA - XBB^TX + C^TC = 0 \ . \qquad (5.10)$$

The solution to this problem was numerically computed on a UNIVAC 1100/83 in double precision using RICPACK. The machine precision is near 10^{-18} in this case. The results of interest are summarized in Table 5.3. Although an exact hand solution was not possible in this case, the behavior of the residual can be used to judge the solution accuracy. This example was designed to assess the effect of the separation of the closed-loop spectrum on solution accuracy and the ability of condition estimates to detect degrading accuracy. The column CLP in Table 5.3 indicates the position (real part) of the closed-loop pole nearest the imaginary axis in the complex plane.

TABLE 5.3

Numerical Results for Example 2, $\varepsilon = 10^{-N}$

N	CLP	$\kappa(V_{11})$	$\kappa A_c(X)$ (4.18)	$\kappa B(X)$ (4.4)	r (5.3)	Newton iterations	r (5.3)
0	10^{0}	10^{1}	10^{0}	10^{2}	10^{-16}	1	10^{-16}
3	10^{-6}	10^{0}	10^{6}	10^{7}	10^{-12}	1	10^{-14}
5	10^{-10}	10^{0}	10^{10}	10^{11}	10^{-8}	1	10^{-12}
7	10^{-14}	10^{0}	10^{18}	10^{15}	10^{-8}	2	10^{-16}
8	10^{-16}	10^{0}	10^{16}	10^{17}	10^{-1}	2	10^{-16}
9	10^{-18}	10^{0}	10^{18}	10^{18}	10^{-1}	-	-

One can see from this example that $\kappa(V_{11})$ provides no indication of loss of accuracy in the solution. However, in this example $\kappa A_c(X)$ and $\kappa B(X)$ correlate directly with the behavior of the residual, and thus, the solution accuracy. Ward balancing of the eigenproblem had no

noticeable effect on solution accuracy for a given value of ε. Newton iterations did significantly improve solution accuracy, as measured by the residual, until the condition $\kappa A_c(X) \approx \kappa B(X) \approx 1./(\text{machine precision})$. At this point, the iterations failed to converge to the desired solution, as was the case in example 1.

This example shows that separation of the closed-loop spectrum does indeed influence the numerical accuracy of the Riccati solution and that the condition estimates in which separation is a factor provide good indicators of solution degeneracy.

The following example illustrates the effect of ill-conditioning of the R weighting matrix, with respect to inversion, on the numerical solution for the continuous-time case. Recall that ill-conditioning of R is not necessarily a problem in the discrete-time case since its inverse is not explicitly required.

Example 3

$$\dot{x} = \begin{bmatrix} -.1 & 0 \\ 0 & -.02 \end{bmatrix} x + \begin{bmatrix} .1 & 0 \\ .001 & .01 \end{bmatrix} u$$

$$y = [10. \quad 100.]x \tag{5.11}$$

$$\text{minimize } \int_0^\infty (y^T y + u^T \begin{bmatrix} 1+\varepsilon & 1 \\ 1 & 1 \end{bmatrix} u)dt \tag{5.12}$$

The system is completely controllable and observable. As $\varepsilon \to 0$, the R matrix approaches singularity. The applicable ARE is

$$A^T X + XA - XBR^{-1}B^T X + C^T C = 0 \ . \tag{5.13}$$

The solution to this problem was numerically computed on a UNIVAC 1100/83 in double precision using RICPACK. The machine precision is near 10^{-18} in this case. The results of interest are summarized in Tables 5.4 and 5.5. Ward balancing was employed in the calculations for Table 5.4, and coordinate balancing was employed for Table 5.5. Results of calculations where no balancing was applied were nearly identical to those of Table 5.5 for coordinate balancing.

TABLE 5.4

Numerical Results for Example 3, Ward Balancing, $\varepsilon=10^{-N}$

N	$\kappa(R)$	$\kappa(V_{11})$	$\kappa A_c(X)$ (4.18)	$\kappa B(X)$ (4.4)	r (5.3)	Newton iterations	r (5.3)
0	10^1	10^1	10^0	10^2	10^{-17}	1	10^{-17}
2	10^2	10^2	10^2	10^6	10^{-17}	1	10^{-17}
4	10^4	10^3	10^3	10^{10}	10^{-14}	2	10^{-14}
6	10^6	10^4	10^3	10^{11}	10^{-12}	10	10^{-12}
8	10^8	10^5	10^3	10^{13}	10^{-10}	10	10^{-10}
10	10^{10}	10^6	10^3	10^{15}	10^{-8}	10	10^{-8}
12	10^{12}	10^7	10^3	10^{17}	10^{-6}	10	10^{-8}
14	10^{14}	10^8	10^3	10^{19}	10^{-5}	10	10^{-5}
16	10^{16}	10^9	10^3	10^{21}	10^{-3}	10	10^{-3}

TABLE 5.5

Numerical Results for Example 3, System Balancing, $\epsilon=10^{-N}$

N	$\kappa(R)$	$\kappa(V_{11})$	$\kappa A_c(X)$ (4.18)	$\kappa B(X)$ (4.4)	r (5.3)	Newton iterations	r (5.3)
0	10^1	10^3	10^0	10^3	10^{-15}	2	10^{-17}
2	10^2	10^3	10^2	10^6	10^{-15}	2	10^{-17}
4	10^4	10^3	10^3	10^9	10^{-12}	7	10^{-14}
6	10^6	10^3	10^3	10^{11}	10^{-12}	10	10^{-12}
8	10^8	10^3	10^3	10^{13}	10^{-9}	10	10^{-11}
10	10^{10}	10^3	10^3	10^{15}	10^{-6}	10	10^{-9}
12	10^{12}	10^3	10^3	10^{17}	10^{-6}	10	10^{-6}
14	10^{14}	10^3	10^3	10^{19}	10^{-2}	10	10^{-4}
16	10^{16}	10^3	10^3	10^{21}	10^{-1}	10	10^{-2}

The data in Table 5.4 indicate that $\kappa(R)$ with respect to inversion accurately reflects the behavior of the residual. Also, $\kappa(V_{11})$ with respect to inversion is too optimistic in its estimation of the problem condition and $\kappa B(X)$ is too pessimistic. $\kappa A_c(X)$ provides no information in this case. A maximum of 10 Newton iterations was allowed, and where 10 appears in the table, convergence did not occur. The value for the residual in that case is the residual associated with the solution at the tenth iteration. One can see that the ill-conditioning of R begins to dominate the numerical accuracy when $\kappa(R) > 10^6$. Since R^{-1} is involved in the Newton iteration calculations, iterative improvement does not improve accuracy when $\kappa(R)$ dominates. The

convergence criteria used for the Newton iterations do not recognize this fact and stop the iterations.

The data in Table 5.5 are essentially the same as that in Table 5.4 except for one important aspect. $\kappa(V_{11})$ with respect to inversion provides no information on the problem conditioning once $\kappa(R) \geq \kappa(V_{11})$. This may indicate that Ward balancing will cause other sources of problem ill-conditioning, beside unstabilizability of the model and singularity of E, to be reflected in $\kappa(V_{11})$.

The preceding examples illustrate that none of the potential measures of conditioning are reliable indicators by themselves. However, numerical experience to date has shown that in all cases at least one of the measures will detect the degeneracy of numerical accuracy as it occurs.

CHAPTER 6

SECOND-ORDER MODELS

The behavior of many physical systems in engineering can be modeled by the following system of equations:

$$M\ddot{x}(t) + (D + G)\dot{x}(t) + Kx(t) = f(t) \tag{6.1}$$

where $x,f \in R^n$ and M,D,G and $K \in R^{nxn}$. Moreover, in describing physical systems one can make the following assumptions without loss of generality:

$$\begin{aligned} M &= M^T > 0, && \text{generalized capacitive storage} \\ D &= D^T \geq 0, && \text{generalized energy dissipators} \\ G &= -G^T, && \text{generalized conservative elements} \\ K &= K^T \geq 0, && \text{generalized inductive storage.} \end{aligned} \tag{6.2}$$

The model (6.1) can describe electrical, mechanical, thermal, and other systems by appropriate choice of "through" variables, f(current in electrical systems, force in mechanical systems, etc.) and "across" variables, x(voltage, displacement, etc.) [48]. Analogies exist among the various types of systems.

The model (6.1) can result directly from lumped parameter models, or finite approximations to distributed parameter systems described by partial differential equations. One large class of systems of current importance are large space structures (LSS), which are large distributed parameter systems that are most often discretized by the finite element method into the form (6.1) [49]. The problem of controlling LSS motivated the studies of this chapter, although the results are applicable to any system described by (6.1). The first section defines

the LSS framework and explores the inherent structure of (6.1) and (6.2) that might be computationally exploitable. The second section considers specifically the solution of the GARE associated with second-order models. In the third section, criteria are discussed for the determination of controllability, stabilizability, observability or detectability of (6.1).

6.1 Second-Order-Model Structure in the LSS Framework

In the LSS framework, x is a displacement vector, f is a force vector, M is the mass matrix, K is the stiffness matrix, D is the damping matrix, and the G matrix gives rise to gyroscopic forces. In general, n is initially of very high order and the aforementioned matrices may be sparse. The force vector f is of the form

$$f(t) = Fu(t) \tag{6.3}$$

where $u \in R^m$ is the control input, and $F \in R^{nxm}$ is the input matrix. One usually considers an output equation of the form

$$y(t) = Px(t) + V\dot{x}(t) \tag{6.4}$$

where

$y \in R^r$ and $P, V \in R^{rxn}$.

The traditional method for dealing with models of the form (6.1) is to transform to the equivalent standard matrix first-order (state variable) form

$$\dot{z}(t) = Az(t) + Bu(t) \tag{6.5}$$

where $z = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$ is the state of dimension 2n. That is, the model (6.1) would be transformed to the following

$$\begin{bmatrix} \dot{x}(t) \\ \ddot{x}(t) \end{bmatrix} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}(D+G) \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ F \end{bmatrix} u(t) \quad . \tag{6.6}$$

Unfortunately, this procedure, although it is conceptually simple, faces practical computational drawbacks. First, the symmetry, definiteness and sparsity structure of the M, D, G and K matrices are not exploited. In general, $M^{-1}K$ and $M^{-1}(D+G)$ are not symmetric and are dense even though M, D, G and K are symmetric and sparse. Moreover, if M is nearly singular, then M^{-1} may be computationally ill-determined.

Consider the following generalized first-order realization of (6.1), (6.3), and (6.4):

$$\begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} + \begin{bmatrix} 0 & I \\ -K & -(D+G) \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ F \end{bmatrix} u$$

$$y = \begin{bmatrix} P & V \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} \tag{6.7}$$

Pre-multiplying (6.7) by $\begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix}^{-1}$ yields the "standard" first-order model (6.6). System (6.7) is of the generalized first-order form

$$E\dot{z} = Az + Bu$$

$$y = Cz \tag{6.8}$$

by appropriate definition of z, A, B, C and E.

Other first-order realizations of this form which are of potential interest include:

$$\begin{bmatrix} D+G & M \\ M & 0 \end{bmatrix} \dot{z} = \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix} z + \begin{bmatrix} F \\ 0 \end{bmatrix} u \tag{6.9}$$

$$\begin{bmatrix} D+G & M \\ -M & 0 \end{bmatrix} \dot{z} = \begin{bmatrix} -K & 0 \\ 0 & -M \end{bmatrix} z + \begin{bmatrix} F \\ 0 \end{bmatrix} u \tag{6.10}$$

$$\begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix} \dot{z} = \begin{bmatrix} 0 & -K \\ -K & -(D+G) \end{bmatrix} z + \begin{bmatrix} 0 \\ F \end{bmatrix} u \tag{6.11}$$

Note that (6.9) and (6.11) have symmetric E and A matrices when G = 0, while (6.1) has skew-symmetric E and symmetric A when D = 0. These properties are computationally advantageous in the problems in the remainder of this chapter.

6.2 The GARE for Second-Order Models

Consider the model (6.7), (6.4) in the context of the continuous-time optimal regulator problem of section 2.3.1 and let $\bar{D} = D+G$. The GARE for this problem from (2.30) is

$$\begin{aligned} 0 = &\begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix} X \begin{bmatrix} 0 & I \\ -K & -\bar{D} \end{bmatrix} + \begin{bmatrix} 0 & -K \\ I & -\bar{D}^T \end{bmatrix} X \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix} + \begin{bmatrix} P^T \\ V^T \end{bmatrix} Q[P \; V] \\ &- \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix} X \begin{bmatrix} 0 \\ F \end{bmatrix} R^{-1}(0 \; F^T) X \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix} \end{aligned} \tag{6.12}$$

and the optimal control law is given by

$$u^0 = -R^{-1}[0 \; F^T] X \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} . \tag{6.13}$$

If one solves the 2n x 2n GARE (6.12) employing the method of section 2.3, then one is faced with a 4n x 4n generalized eigenproblem. Clearly, this is undesirable if n is itself a large number.

Instead we expand $X = X^T \geq 0$ into nxn blocks as [50]

$$\begin{bmatrix} W & Y \\ Y^T & Z \end{bmatrix} \geq 0,\ W = W^T \geq 0,\ WW^+Y = Y,\ Z - Y^TW^+Y \geq 0. \tag{6.14}$$

Substituting (6.14) into (6.12) and multiplying out yields the following three equations

$$0 = -YK - KY^T - YFR^{-1}F^TY^T + P^TQP \tag{6.15}$$

$$0 = -MZ\bar{D} - \bar{D}^TZM - MZFR^{-1}F^TZM + V^TQV + MY^T + YM \tag{6.16}$$

$$0 = W - Y\bar{D} - KZM - YFR^{-1}F^TZM + P^TQV \tag{6.17}$$

We see that (6.15) is an nxn Riccati-type equation from which Y could be obtained. Once Y is determined, (6.16) becomes an nxn GARE from which we can solve for Z. Equation (6.17) then simply defines W.

If only the feedback (6.13) is desired, we see (by substituting (6.14) into (6.13) and expanding) that

$$u^0 = -R^{-1}F^T(Y^Tx + ZM\dot{x}) \quad . \tag{6.18}$$

Therefore, one need not solve (6.17) for W.

It is computationally advantageous to only require the solution of two nxn Riccati equations rather than one 2n x 2n Riccati equation, since the solution of an NxN Riccati equation takes $O(N^3)$ operations. To realize even greater computational advantage, we consider two common special cases. First, we consider the case when there is no damping or gyroscopic forces, i.e., $\bar{D} = 0$. One can see that (6.16) is then no longer a GARE; it reduces to a simpler quadratic equation.

Next, consider the velocity output case, i.e., P = 0. Then, a solution for (6.15) is Y = 0. One can solve the GARE (6.16) for Z and the optimal control becomes

$$u^0 = -R^{-1}F^T Z M \dot{x} \tag{6.19}$$

Obviously, solving one nxn GARE is computationally simpler than solving a 2n x 2n GARE regardless of the size of n. Based on these results, we can state the following theorem.

Theorem 6.1: Given the system

$$M\ddot{x}(t) + \bar{D}\dot{x}(t) + K x(t) = F u(t) \tag{6.20}$$

with output

$$y(t) = V\dot{x}(t) \tag{6.21}$$

and criterion

$$J = \int_0^\infty (y^T Q y + u^T R u)dt. \tag{6.22}$$

Then the unique stabilizing control which minimizes (6.22) is given by

$$u^0(t) = -R^{-1}F^T Z M \dot{x}(t) \tag{6.23}$$

where $Z = Z^T \geq 0$ satisfies the GARE

$$0 = -MZ\bar{D} - \bar{D}^T Z M - MZFR^{-1}F^T Z M + V^T Q V \quad . \tag{6.24}$$

Proof: The theorem is a restatement of the results of this section.

6.3 Controllability and Observability Criteria for Second-Order Models

We now seek to establish controllability and observability criteria for the model (6.1), (6.3) and (6.4). Controllability and observability of this model have been shown to provide important insights into modal behavior of the system and to furnish information

on the number and positioning of sensors and actuators [51], [52]. Also, controllability and observability information can be used in determining which modes to retain when performing model reduction [53].

One could apply the traditional methods for determining controllability and observability to the transformed first-order model (6.6) with state dimensions 2n. Unfortunately, the resulting standard tests (cast in terms of a 2n-th order "A" matrix) do not take advantage of the symmetry, definiteness and sparsity structure of the matrices M, D, G, and K. Also, computational problems may exist if M is near singular or n is very large.

Other conditions have been derived by Hughes and Skelton [51], which exploit the specialized structure of (6.1) and (6.2) for the cases D = 0 and D+G = 0. However, these conditions could suffer from computational difficulties since they require knowledge of the full modal transformation matrix, whose columns are the eigenvectors corresponding to the eigenvalues λ of the generalized eigenproblem

$$[\lambda M + K]x = 0 \tag{6.25}$$

where $\pm\lambda^{1/2}$ is the modal frequencies. An equivalent form of (6.25) is the simple eigenproblem

$$-M^{-1}Kx = \lambda x \tag{6.26}$$

since M is nonsingular. But this form is computationally undesirable for the reasons already mentioned. In addition, computation of the eigenvectors of (6.25) and (6.26) is ill-conditioned whenever the λ is repeated or nearly equal [15], which is often the case in LSS.

The remainder of this section focuses on conditions for controllability (or, more generally, stabilizability) and observability (or, more generally, detectability) which take advantage of the structure of (6.1) and (6.2), but extend the results of [51] and are computationally more tractable. Most of the computational attractiveness of the new criteria accrue from the fact that an initial modal transformation is not necessary. Thus, if just a few "important" modes are known--and there exist techniques to determine just selected modes, e.g., [54], [55]--these modes can be tested for, say controllability, by a test involving just the model matrices M, D, G, K, and F.

Definition 6.1:

$$\Omega := \left\{ \lambda_i \quad : \lambda_i \text{ is a generalized eigenvalue of the problem } \begin{bmatrix} 0 & I \\ -K & -(D+G) \end{bmatrix} - \lambda \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix} \right\} \tag{6.27}$$

$$:= \{\text{modes of the system (6.7)}\}$$

$$\text{Also, } \Omega^+ := \{\lambda \in \Omega : \operatorname{Re} \lambda \geq 0\} \tag{6.28}$$

$$\Omega^- := \{\lambda \in \Omega : \operatorname{Re} \lambda < 0\} \tag{6.29}$$

For M nonsingular there are 2n modes, and Ω^+ and Ω^- are the sets of the unstable and stable modes, respectively. Alternatively, Ω can be determined in terms of the generalized eigenvalue problems

$$\begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix} - \lambda \begin{bmatrix} D+G & M \\ M & 0 \end{bmatrix} \tag{6.30}$$

$$\begin{bmatrix} -K & 0 \\ 0 & -M \end{bmatrix} - \lambda \begin{bmatrix} D+G & M \\ -M & 0 \end{bmatrix} \tag{6.31}$$

$$\begin{bmatrix} 0 & -K \\ -K & -(D+G) \end{bmatrix} - \lambda \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix} \tag{6.32}$$

whichever yields the greatest computational advantage. Computation of eigenvalues for problems of the form (6.32) is discussed in [54] for the case G = 0.

Theorem 6.2: (Hautus [56]) The system

$$\dot{x} = Ax + Bu \; ; \quad x(t) \in R^n \tag{6.33}$$

is

a) Controllable (stabilizable) if and only if

$$\text{rank } [A - \lambda I, \; B] = n; \text{ for all } \lambda \in \Lambda(A) \; (\Lambda^+(A)) \tag{6.34}$$

b) Observable (detectable) if and only if

$$\text{rank} \begin{bmatrix} C \\ (A - \lambda I) \end{bmatrix} = n \; ; \text{ for all } \lambda \in \Lambda(A) \quad (\Lambda^+(A)), \tag{6.35}$$

where $\Lambda(A) := \{\lambda: (A- \lambda I)x = 0, \; x \neq 0\}$

$$:= \text{Spectrum of A} \tag{6.36}$$

Proof: [56]

Clearly then, the system (6.8) with E nonsingular is controllable (stabilizable) if and only if

$$\text{rank } [A - \lambda E, \; B] = 2n \text{ for all } \lambda \in \Lambda(E^{-1}A) \; (\Lambda^+(E^{-1}A)) \tag{6.37}$$

and observable (detectable) if and only if

$$\text{rank} \begin{bmatrix} C \\ A - \lambda E \end{bmatrix} = 2n \text{ for all } \lambda \in \Lambda(E^{-1}A) \; (\Lambda^+(E^{-1}A)) \tag{6.38}$$

We now exploit the structure of A, B, C to derive controllability, etc., criteria directly in terms of M, D, G, K, F, P, and V.

Theorem 6.3: The system (6.1), (6.3) is controllable (stabilizable) if and only if

$$\text{rank } [\lambda^2 M + \lambda(D+G) + K, \; F] = n; \text{ for all } \lambda \in \Omega \; (\Omega^+) \tag{6.39}$$

Proof: By Theorem 6.2, the system (6.1), (6.3) is controllable (stabilizable) if and only if

$$2n = \text{rank}\,[A - \lambda E,\ B];\ \text{for all } \lambda \in \Omega\ (\Omega^+)$$

$$= \text{rank}\begin{bmatrix} -\lambda I & I & 0 \\ -K & -D-G-\lambda M & F \end{bmatrix} \quad \text{from (6.7) and (6.8)}$$

$$= \text{rank}\begin{bmatrix} \lambda M+D+G & I \\ I & 0 \end{bmatrix}\begin{bmatrix} -\lambda I & I & 0 \\ -K & -D-G-\lambda M & F \end{bmatrix}\begin{bmatrix} -I & 0 & 0 \\ -\lambda I & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

$$= \text{rank}\begin{bmatrix} \lambda^2 M+\lambda(D+G)+K & 0 & F \\ 0 & I & 0 \end{bmatrix}$$

Clearly this obtains if and only if

$$\text{rank}\,[\lambda^2 M+\lambda(D+G)+K,\ F] = n \text{ for all } \lambda \in \Omega(\Omega^+).$$

Note that λ is a scalar, so that sparsity in the problem is preserved. Also, no inverses and no initial transformations are necessary. Finally, note that each mode of the system can be checked individually without transforming the system to modal coordinates.

Theorem 6.4: The system (6.1), (6.4) is observable (detectable) if and only if

$$\text{rank}\begin{bmatrix} \lambda V+P \\ \lambda^2 M+\lambda(D+G)+K \end{bmatrix} = n \text{ for all } \lambda \in \Omega\ (\Omega^+). \tag{6.40}$$

Proof: By Theorem 6.2 the system (6.1), (6.4) is observable (detectable) if and only if

$$2n = \text{rank} \begin{bmatrix} C \\ A - \lambda E \end{bmatrix}; \text{ for all } \lambda \in \Omega\ (\Omega^+)$$

$$= \text{rank} \begin{bmatrix} P & V \\ -\lambda I & I \\ -K & -D-G-\lambda M \end{bmatrix} \text{ from (6.7) and (6.8)}$$

$$= \text{rank} \begin{bmatrix} I & -V & 0 \\ 0 & -\lambda M-D-G & -I \\ 0 & I & 0 \end{bmatrix} \begin{bmatrix} P & V \\ -\lambda I & I \\ -K & -D-G-\lambda M \end{bmatrix} \begin{bmatrix} I & 0 \\ \lambda I & I \end{bmatrix}$$

$$= \text{rank} \begin{bmatrix} \lambda V+P & 0 \\ \lambda^2 M+\lambda(D+G)+K & 0 \\ 0 & I \end{bmatrix}$$

Clearly, this obtains if and only if

$$\text{rank} \begin{bmatrix} \lambda V+P \\ \lambda^2 M+\lambda(D+G)+K \end{bmatrix} = n \text{ for all } \lambda \in \Omega\ (\Omega^+).$$

Note that an alternative proof of Theorems 6.3 and 6.4 is to observe that the results are essentially restatements of the spectral criteria for left or right coprimeness of the appropriate polynomial matrices (quadratic, in this case). However, we have exploited here the numerically useful characterization of Ω in Definition 6.1, and the proofs are direct and require no polynomial matrix theory. Several special cases of Theorems 6.3 and 6.4 are of interest in many systems and are now stated as corollaries.

Corollary 6.3.1: When $D+G = 0$ (i.e., no damping or gyroscopic forces), the system (6.1), (6.3) is controllable if and only if

$$\text{rank} \left[-\omega_i^2 M+K,\ F\right] = n \ ; \ i = 1,\ldots,n \tag{6.41}$$

where $\omega_i = \sqrt{\lambda_i}$; $\lambda_i \in \Omega(M^{-1}K)$ (Note: $\lambda_i \geq 0$).

Proof: When $D+G = 0$

$$\Omega = \{\pm j\omega_i \ ; \ i = 1,\ldots,n\}$$

and (6.41) follows directly from Theorem 6.3.

Corollary 6.3.2: When D+G = 0 and the system (6.1), (6.3) is in modal form, then (6.1), (6.3) is controllable if and only if

$$\text{rank } F_r = n_r; \ (r = 1,\ldots,R) \tag{6.42}$$

where F_r are partitioned rows of the modally transformed F matrix corresponding to the multiplicities n_i of the ω_i; $n_1 + \ldots + n_R = n$, and R is the number of distinct ω_i. This is Theorem 1 of [51].

Proof: In modal form

$M = I$, $K = \text{diag}[\omega_1^2,\ldots\omega^2]$, F_n = modally transformed F matrix. Then (6.42) follows directly from Corollary 6.3.1.

Corollary 6.3.3: When K = 0 the system (6.1), (6.3) is controllable if and only if rank F = n.

Proof: When $K = 0$, $\lambda = 0 \in \Omega$ and the corollary follows directly from Theorem 6.3.

Similarly, we can state

Corollary 6.4.1: When D+G = 0 the system (6.1), (6.4) is observable if and only if

$$\text{rank} \begin{bmatrix} j\omega_i V+P \\ -\omega_i^2 M+K \end{bmatrix} = n \ ; \quad i = 1,\ldots,n. \tag{6.43}$$

Proof: This result follows directly from the proof of Corollary 6.3.1 and Theorem 6.4.

Corollary 6.4.2: When D+G = 0 and V = 0 (i.e., no rate feedback) the system (6.1), (6.4) is observable if and only if

$$\text{rank} \begin{bmatrix} P \\ -\omega_i^2 M+K \end{bmatrix} = n; \quad i = 1,\ldots,n. \tag{6.44}$$

Proof: Follows directly from Corollary 6.4.1.

Corollary 6.4.3: When D+G = 0 and the system (6.1), (6.4) is in modal form, then (6.1), (6.4) is observable if and only if

$$\text{rank}[j\omega_i V_r + P_r] = n_r; \quad (r = 1,\ldots, R) \tag{6.45}$$

where $[j\omega_i V_r + P_r]$ are the suitably partitioned columns of the modally transformed $[j\omega_i V + P]$ matrix.

Proof: In modal form M = I, K = $\text{diag}[\omega_1^2,\ldots,\omega_n^2]$, $[j\omega_i V + P]$ = modally transformed $[j\omega_i V + P]$ matrix and (6.45) follows directly from (6.43).

Corollary 6.4.4: When D+G = 0, the system (6.1), (6.4) is in modal form, and V = 0 then (6.1), (6.4) is observable if and only if

$$\text{rank } P_r = n_r; \quad (r = 1,\ldots,R). \tag{6.46}$$

Proof: Follows directly from Corollary 6.4.3

Corollary 6.4.5: When K = 0 the system (6.1), (6.4) is observable if and only if rank P = n.

Proof: Set w_i = 0 and K = 0 in (6.43).

Note that the above Corollaries have obvious analogues in terms of stabilizability and detectability, as appropriate.

CHAPTER 7

CONCLUSION

We have examined some numerical issues regarding the solution of a very general form of the algebraic Riccati equation in both the continuous- and discrete-time formulations. These generalized equations resulted from control and filtering problems for systems in generalized state space form with performance criteria that included cross-coupling between the state and input. The basic solution method considered was the Schur technique. The Schur technique was preferred because of its good numerical properties, especially when some closed-loop-system eigenvalues were closely spaced. The generalized eigenproblem framework employed allows solutions when E and R are ill-conditioned with respect to inversion without undue influence of the ill-conditioning on the solution process. Singular R matrices were permissible in the discrete-time case.

A Newton-type iterative refinement procedure for the Riccati solution was derived. In the most general case, it required solution of a Sylvester equation at each iteration. Numerical results indicated that the iterative refinement improved numerical accuracy significantly in all cases where accuracy was lost due to ill-conditioning of the Riccati problem except when R was ill-conditioned with respect to inversion in the continuous-time problem. Lack of improvement in this case was attributed to the fact that the iterative procedure required R^{-1} explicitly at each iteration.

Condition estimates for the Riccati problem were examined. New condition estimates were also derived. The behavior of these estimates and their ability to detect degraded accuracy of the Riccati solution were evaluated in numerical examples. All numerical examples employed the software package RICPACK for the solution of the Riccati equation. RICPACK was developed as a research tool to aid in the study of numerical issues related to the solution of algebraic Riccati equations. It was found that no single condition measure considered herein would reliably reflect the accuracy of the Riccati solution in all cases. However, ill-conditioning was always detected by at least one of them.

The special structure of linear system models in second-order form was considered. Computational advantage was sought in solving Riccati equations related to these models. Special computational advantage was realized in the velocity feedback control problem. Controllability and observability tests for second-order models were derived directly in terms of the system model matrices. These tests had the additional advantage of enabling one to test individual model modes.

Some topics of continuing research in these areas will include:

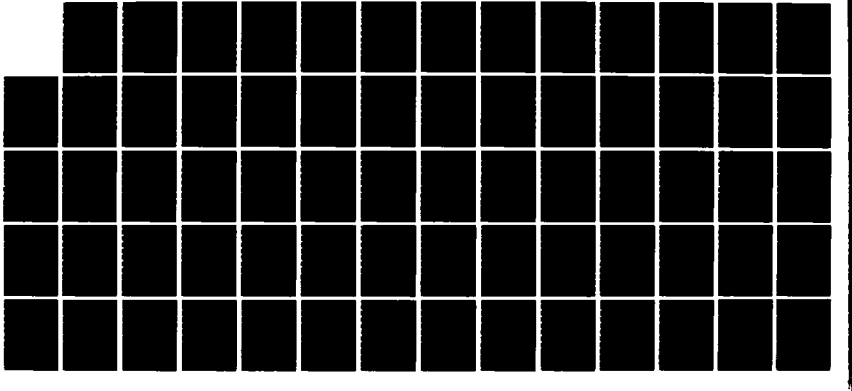
1. Iterative refinement procedure in the continuous-time formulation that does not require explicit inversion of the R matrix.
2. Further development of condition estimates for the Riccati problem, especially for the case $E \neq I$ and considering other forms for the perturbation δF.

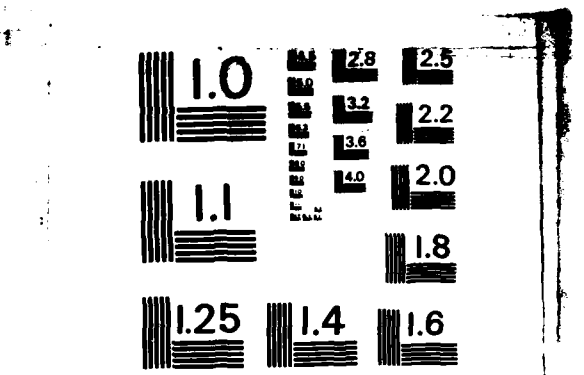
3. Further exploitation of the special structure of models in second-order form in all types of control and filtering computations.

We note here that all further research in the area of numerically solving algebraic Riccati equations can benefit from the availability of RICPACK.

AD-A139 929 NUMERICAL SOLUTION OF ALGEBRAIC MATRIX RICCATI EQUATIONS(U) NAVAL WEAPONS CENTER CHINA LAKE CA W F ARNOLD FEB 84 NWC-TP-6521 SBI-AD-E900 331 2/2

UNCLASSIFIED F/G 12/1 NL





REFERENCES

1. H. Kwakernaak and R. Sivan. *Linear Optimal Control Systems*. Wiley Interscience, New York, 1972.

2. B. D. O. Anderson and J. B. Moore. *Optimal Filtering*. Prentice-Hall, Englewood Cliffs, New Jersey, 1979.

3. L. M. Silverman. "Discrete Riccati Equations: Alternative Algorithms, Asymptotic Properties, and System Theory Interpretations," in *Advances in Control Systems*, Vol. 12, Leondes, Ed., Academic Press, New York, 1976.

4. W. T. Reid. *Riccati Differential Equations*. Academic Press, New York, 1972.

5. K. Martensson. *New Approaches to the Numerical Solution of Optimal Control Problems*. Lund Institute of Technology, Report 7206, March 1972.

6. A. J. Laub. "A Schur Method for Solving Algebraic Riccati Equations," *IEEE Trans. Automatic Control*, Vol. AC-24, pp. 913-921, December 1979.

7. A. J. Laub. *A Schur Method for Solving Algebraic Riccati Equations*. Lab. for Inform. and Decision Systems, MIT, LIDS Rep. LIDS-R-859, 1978.

8. T. Pappas, A. J. Laub, and N. R. Sandell. "On the Numerical Solution of the Discrete-Time Algebraic Riccati Equation," *IEEE Trans. Automatic Control*, Vol. AC-25, pp. 631-641, August 1980.

9. A. Enami-Naeini. *Application of the Generalized Eigenstructure Problem to Multivariable Systems and the Robust Servomechanism for a Plant which Contains an Implicit Internal Model*. Ph.D. Thesis, Stanford University, April 1981.

10. P. Van Dooren. "A Generalized Eigenvalue Approach for Solving Riccati Equations." *SIAM J. Sci. Stat. Comput.*, Vol. 2, pp. 121-135, 1981.

11. A. J. Laub. "Schur Techniques in Invariant Imbedding Methods for Solving Two-Point Boundary Value Problems," *Proc. 21st IEEE CDC*, pp. 56-61, 1982.

12. K. H. Lee. *Generalized Eigenproblem Structures and Solution Methods for Riccati Equations*. Ph.D. Thesis, University of Southern California, January 1983.

13. D. L. Kleinman. "On an Iterative Technique for Riccati Equation Computations," *IEEE Trans. Automatic Control*, Vol. 13, pp. 114-115, February 1968.

14. G. A. Hewer. "An Iterative Technique for the Computation of the Steady State Gains for the Discrete Optimal Regulator," *IEEE Trans. Automatic Control*, Vol. AC-16, pp. 382-384, August 1971.

15. G. W. Stewart. *Introduction to Matrix Computations*. Academic Press, New York, 1973.

16. J. R. Rice. *Matrix Computations and Mathematical Software*. McGraw-Hill, New York. 1981.

17. G. H. Golub and J. H. Wilkinson. "Ill-conditioned Eigensystems and the Computation of the Jordan Canonical Form." *SIAM Review*, Vol. 18, pp. 578-619, 1976.

18. J. H. Wilkinson. *The Algebraic Eigenvalue Problem*. Oxford University Press, London, 1965.

19. G. W. Stewart. "On the Sensitivity of the Eigenvalue Problem $Ax=\lambda Bx$," *SIAM J. Numer. Anal.*, Vol. 9, pp. 669-686, December 1972.

20. R. Byers. "Hamiltonian and Symplectic Algorithms for the Algebraic Riccati Equation." Ph.D. Thesis, Cornell University, January 1983.

21. J. R. Rice. "A Theory of Condition." *SIAM J. Numer. Anal.*, Vol. 3, pp. 287-310, 1966.

22. J. H. Wilkinson. *Rounding Errors in Algebraic Processes*. Prentice-Hall, Englewood Cliffs, New Jersey, 1963.

23. A. P. Sage. *Optimum Systems Control*. Prentice-Hall, Englewood Cliffs, New Jersey, 1968.

24. A. Emami-Naeini and G. F. Franklin. "Deadbeat Control and Tracking of Discrete-Time Systems." *IEEE Trans. Automatic Control*, Vol. AC-27, pp. 176-181, February 1982.

25. N. R. Sandell. "On Newton's Method for Riccati Equation Solution." *IEEE Trans. Automatic Control*, Vol. AC-19, pp. 254-255, June 1974.

26. F. A. Ferrar and R. C. DiPietro. *Comparative Evaluation of Numerical Methods for Solving the Algebraic Matrix Riccati Equation.* United Technologies Research Center, East Hartford, CT, Rep. R76-140268-1, December 1976.

27. C. W. Merriam III. *Automated Design of Control Systems.* Gordon and Breach Science Publishers, New York, 1974.

28. D. Cobb. "Descriptor Variable Systems and Optimal State Regulation." *IEEE Trans. Automatic Control*, Vol. AC-28, pp. 601-611, May 1983.

29. R. S. Bucy. "The Riccati Equation and Its Bounds." *J. of Computer and System Sciences*, Vol. 6, pp. 343-353, 1972.

30. J. C. Allwright. "A Lower Bound for the Solution of the Algebraic Riccati Equation of Optimal Control and a Geometric Convergence Rate for the Kleinman Algorithm." *IEEE Trans. Automatic Control*, Vol. AC-25, pp. 826-829, August 1980.

31. W. H. Kwon and A. E. Pearson. "A Note on the Algebraic Riccati Equation." *IEEE Trans. Automatic Control*, Vol. AC-22, pp. 143-144, February 1977.

32. K. V. Patel and M. Toda. "On Norm Bounds for Algebraic Riccati and Lyapunov Equations." *IEEE Trans. Automatic Control*, Vol. AC-23, pp. 87-88, February 1978.

33. G. Lanholz. "A New Lower Bound on the Cost of Optimal Regulators." *IEEE Trans. Automatic Control*, Vol. AC-24, pp. 353-354, April 1979.

34. K. Yasuda and K. Hirai. "Upper and Lower Bounds on the Solution of the Algebraic Riccati Equation." *IEEE Trans. Automatic Control*, Vol. AC-24, pp. 483-487, June 1979.

35. G. W. Stewart. "Error Bounds for Approximate Invariant Subspaces of Closed Linear Operators." *SIAM J. Numer. Anal.*, Vol. 8, pp. 796-808, December 1971.

36. G. W. Stewart. "Error and Perturbation Bounds for Subspaces Associated with Certain Eigenvalue Problems." *SIAM Review*, Vol. 15, pp. 727-764, October 1973.

37. C. Paige and C. Van Loan. "A Schur Decomposition for Hamiltonian Matrices." *Linear Algebra and Its Applications*, Vol. 41, pp. 11-32, 1981.

38. G. W. Stewart. "On the Perturbation of Pseudo-Inverses, Projections and Linear Least Squares Problems," *SIAM Review*, Vol. 19, pp. 634-662, 1977.

39. J. J. Dongarra et. al. *LINPACK User's Guide*, SIAM, 1979.

40. B. T. Smith, et. al. *Matrix Eigensystem Routines-EISPACK Guide*, Lecture Notes in Computer Science, Vol. 6, Springer-Verlag, 1976.

41. C. B. Moler and G. W. Stewart. "An Algorithm for Generalized Matrix Eigenvalue Problems," *SIAM J. Numer. Anal.*, Vol. 10, pp. 241-256, 1973.

42. R. C. Ward. "Balancing the Generalized Eigenvalue Problem," *SIAM J. Sci. Stat. Comput.*, Vol. 2, pp. 141-152, 1981.

43. B. C. Moore. "Principal Component Analysis in Linear Systems: Controllability, Observability and Model Reduction," *IEEE Trans. Automatic Control*, Vol. AC-26, pp. 17-32, February 1981.

44. P. Van Dooren. "ALGORITHM 590 DSUBSP and EXCHQZ: FORTRAN Subroutines for Computing Deflating Subspaces with Specified Spectrum," *ACM Trans. Math. Soft.*, Vol. 8, pp. 376-382, 1982.

45. A. J. Laub. "On Computing 'Balancing' Transformations," *Proceedings 1980 JACC*, San Francisco, CA, 1980.

46. B. S. Garbow, et. al. *Matrix Eigensystem Routines-EISPACK Guide Extension*, Lecture Notes in Computer Science, Vol. 51, Springer-Verlag, 1977.

47. R. H. Bartels and G. W. Stewart. "Solution of the Matrix Equation AX+XB=C," Algorithm 432, *Comm. ACM*, Vol. 15, pp. 820-826, 1972.

48. R. C. Dorf. *Modern Control Systems*, Second Ed., Addison-Wesley, Reading, MA, 1974.

49. M. J. Balas. "Trends in Large Space Structure Theory: Fondest Hopes, Wildest Dreams," *IEEE Trans. Automatic Control*, Vol. AC-27, pp. 522-535, June 1982.

50. A. Albert. *Regression and the Moore-Penrose Pseudo-Inverse*, Academic Press, New York, 1972.

51. P. C. Hughes and R. E. Skelton. "Controllability and Observability of Linear Matrix-Second-Order Systems," *ASME J. Appl. Mech.*, Vol. 47, pp. 415-420, 1980.

52. M. J. Balas. "Feedback Control of Flexible Systems," *IEEE Trans. Automatic Control*, Vol. AC-23, pp. 674-679, 1978.

53. R. E. Skelton. "Observability Measures and Performance Sensitivity in the Model Reduction Problem," *Int. J. Control*, Vol. 29, pp. 541-556, 1979.

54. D. S. Scott and R. C. Ward. "Solving Symmetric-Definite, Quadratic λ-Matrix Problems without Factorization," *SIAM J. Sci. Stat. Comput.*, Vol. 3, pp. 58-67, 1982.

55. B. N. Parlett. *How to Solve (K-λM)z=0 for Large K and M.* Center for Pure and Applied Math. Tech. Report No. PAM-39, University of California, Berkeley, CA, 1981.

56. N. L. J. Hautus. "Controllability and Observability Conditions of Linear Autonomous Systems," *Proc. Kon. Ned. Akad. Wetensch.*, Ser. A, Vol. 72, pp. 443-448, 1969.

Appendix A

SOFTWARE DESCRIPTION

The purpose of this appendix is to provide more detailed information on the software package RICPACK than is appropriate for the main body of the thesis. However, it is not a complete documentation package for RICPACK. First there will be a discussion of the hierarchy of the software routines employed in the package. Then a sample terminal session for a particular example problem is illustrated. Finally, some appropriate software listings are included.

Figure A.1 illustrates the hierarchy of the software routines. That is, the routines on the upper levels employ the routines of the lower levels. At the lowest level we have the basic matrix manipulation routines like add, subtract, multiply, etc., and some simple combinations of these basic operations. The next level consists of standard routines for linear equations, eigenvalues and singular value decomposition (SVD). Most routines in this level are from LINPACK or EISPACK, or are slight modifications to routines from LINPACK and EISPACK. Subroutines that are modified have the modifications noted in the comment documentation included in the subroutine. A list of the Level 0 and 1 routines is given in Table A.1. The SVD routine is listed separately with the BLAS routines that it requires from LINPACK, as it is the only routine to require the BLAS and could be modified to eliminate said BLAS. No further documentation of these routines is provided in this appendix.

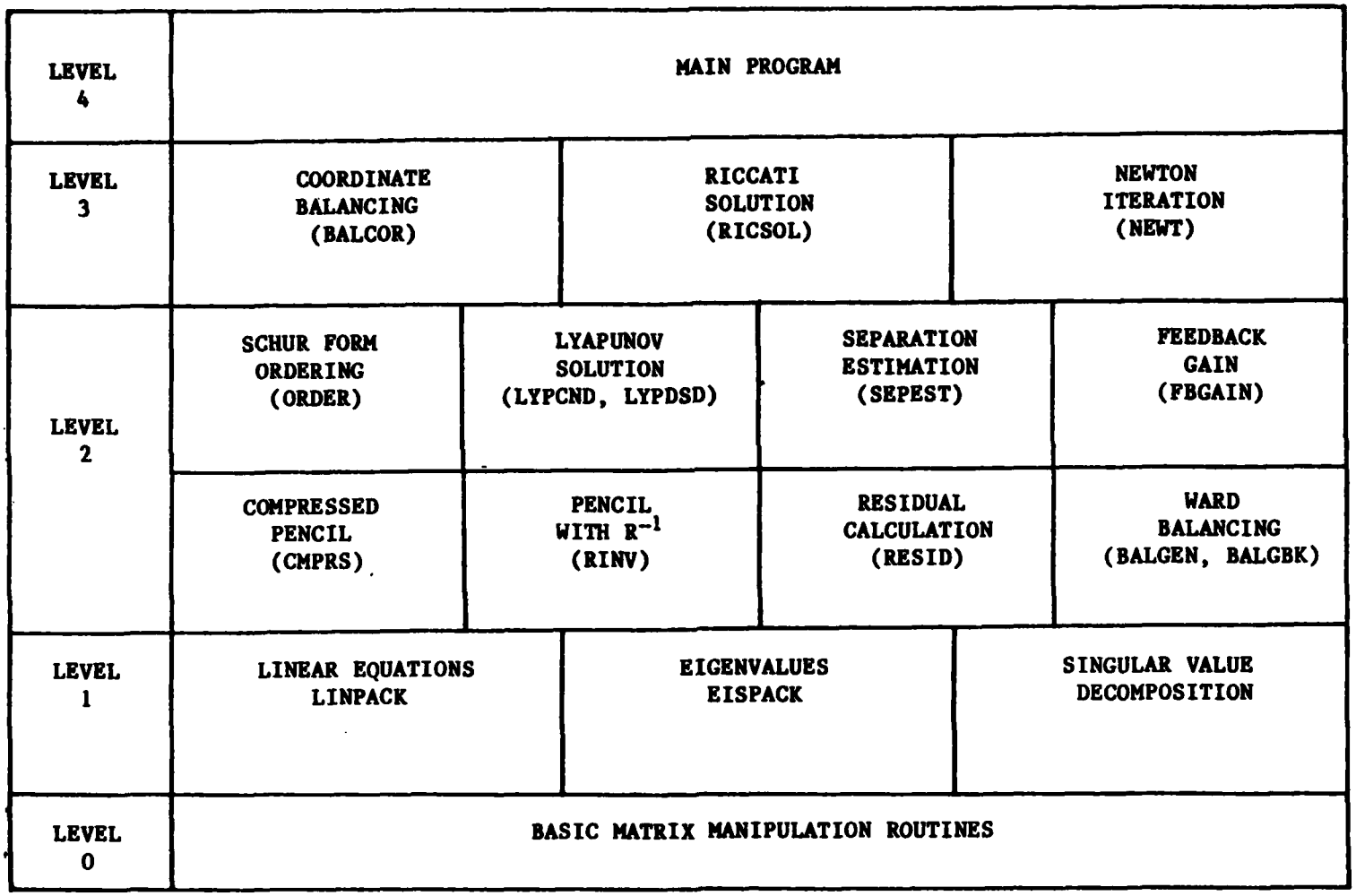


FIGURE A.1. Hierarchy of subroutines in software package RICPACK.

TABLE A.1

Subroutine List for Levels 0 and 1

Level 0	Level 1		DSVDC
BCORBK	BALANC	MLINEQ	DAXPY
D1NRM		ORTHES	DDOT
MADD	DGECOM	ORTRAN	DNRM2
MMUL	DGEFAM	QZHESW	DROT
MOUT	DGESLM	QZITW	DROTG
MQF		QZVAL	DSCAL
MQFWO	DSTSLV	REBAKB	DSWAP
MSCALE	ELMHES	REDUCE	
MSUB	GIV	REDUC2	
MULA	GRADBK	ROTC	
MULB	GRADEQ	ROTR	
MULWOA	HQR	SCALBK	
MULWOB	HQRORT	SCALEG	
PERMUT	IMTQL2	SYMSLV	
SAVE	LINEQ	TRED2	
SEQUIV			
TRNATA			
TRNATB			

Subroutines at Levels 2 and 3 perform more specialized tasks. The task title is given in Figure A.1, and the main subroutine performing the task is shown in parenthesis. The major comment documentation from these main subroutines, as it appears in the software, is given in this appendix. This documentation can be consulted for more details on the purpose of each routine, or for a list of subroutines that each of these main subroutines may call.

Of course, the main driver program is at the highest level (4), and a complete listing is provided. This main program would be rewritten, or at least extensively modified for most applications. This driver was written as a research tool and as such performs calculations not relevant to many analysis and design applications. A higher level language would be more appropriate to interface the Level 0 through Level 3 subroutines with a larger CACSD package and perform the necessary input and output.

A sample output listing from an actual terminal session to solve one particular problem is given in Figure A.2. The problem is Example 1, unbalanced case, for $\epsilon = .0001$. The session illustrates most of the features and options of RICPACK. The computer prompt for data is the ">" sign. The listing is self-explanatory and will not be elaborated further.

The main program listing and heading documentation comment listings for the main subroutines of Levels 2 and 3 follow Figure A.2.

```
@XQT RICPACK.

GENERALIZED ALGEBRAIC MATRIX RICCATI EQUATION SOLVER

ENTER SYSTEM ORDER 'N';  = FOR CONTINUOUS TIME PROBLEMS, - FOR DISCRETE
MAXIMUM ORDER = 20

>2

N = 2


ENTER NUMBER OF SYSTEM INPUTS 'M'
AND NUMBER OF SYSTEM OUTPUTS "L"

>1,1

M = 1     L = 1


ENTER FLAG FOR DESIRED SOLUTION:
-1 FOR STABILIZING SOLUTION
0 FOR ANY SOLUTION
+ 1 FOR DESTABILIZING SOLUTION

>-1

IORD = -1


ENTER BALANCING FLAG:        0 FOR WARD BALANCING
                             1 FOR CO-ORDINATE BALANCING
                             2 FOR NO BALANCING

>2

IBAL = 2

NO BALANCING ATTEMPTED


DO YOU WISH TO ITERATE FOR ROBUSTNESS RECOVERY (Y OR N)

>N

KFLAG = N


DO YOU WISH TO ENTER AN "E" MATRIX (Y OR N)
DEFAULT IS E = IDENTITY MATRIX
```

FIGURE A.2 Sample terminal session output listing.

```
>N

EFLAG = N

USING DEFAULT: "E" = IDENTITY


ENTER THE  2 X  2 SYSTEM MATRIX "A" BY ROWS

>1,0
>0,-2

THE "A" MATRIX IS:

1.00000000000000000    .00000000000000000
.00000000000000000    -2.00000000000000000


ENTER THE  2 X  1 INPUT MATRIX "B" BY ROWS

>.0001
>0

THE "B" MATRIX IS:

.100000000000000000-003
.00000000000000000


ENTER THE  1 X  2 OUTPUT MATRIX "C" BY ROWS

>1,1

THE "C" MATRIX IS:

1.00000000000000000    1.00000000000000000


DO YOU WISH TO ENTER A CONTROL WEIGHTING MATRIX "R" (Y OR N)
DEFAULT IS R = IDENTITY MATRIX

>N

RFLAG = N

USING DEFAULT: "R" = IDENTITY


DO YOU WISH TO ENTER AN OUTPUT WEIGHTING MATRIX "Q" (Y OR N)

>N
```

FIGURE A.2 Sample terminal session output listing (continued).

```
QFLAG = N

USING DEFAULT: "Q" = IDENTITY


DO YOU WISH TO ENTER A STATE/INPUT CROSS-WEIGHTING MATRIX "S" (Y OR N)
DEFAULT IS S = ZERO MATRIX


>N

SFLAG = N

USING DEFAULT: "S" = ZERO MATRIX


THE CLOSED LOOP EIGENVALUES FOR THE STABILIZING RICCATI SOLUTION ARE:

(FOR CONTINUOUS TIME)

-1.00000000000000000    .000000000000000000
-2.00000000000000000    .000000000000000000

THE STABILIZING RICCATI SOLUTION IS:

200000000.512931183    .333333332798460259
.333333332871560127    .249999999722222220

X1N = .20000000 + 001   H!N = .20000000 + 001   K(Z11) = .19402850 + 009

KB(X) = .30000000 + 001       KA(X) = .50000000-008
SEPEST = .20000000 + 001       SEP = .20000000 + 001


DO YOU WISH A RESIDUAL CALCULATION (Y OR N)

>Y

RSFLG = Y

RESIDUAL 1-NORM/SOLUTION 1-NORM =  .129311838731697373-009


DO YOU WANT TO SEE THE RESIDUAL MATRIX (Y OR N)

>Y

RMFLG = Y
```

FIGURE A.2 Sample terminal session output listing (continued).

```
THE RESIDUAL MATRIX, BY ROWS:

-.258623678237199783-001  -.320515168225621316-010
-.320515168225621316-010  .789603050697242101-017


DO YOU WANT TO TRY ITERATIVE IMPROVEMENT (Y OR N)

>Y

NFLAG = Y


ENTER MAXIMUM NUMBER OF NEWTON ITERATIONS

>5

MAX. NO. OF ITERATIONS =  5

CONVERGED AFTER  2 ITERATIONS OF NEWTONS METHOD

CONVERGENCE CRITERIA 1

THE REFINED STABILIZING SOLUTION IS:

200000000.499999999    .333333332777777778
.333333332777777778    .249999999722222223

THE CLOSED LOOP EIGENVALUES FOR THE STABILIZING RICCATI SOLUTION ARE:

-1.00000000499999998    .000000000000000000
-2.00000000000000000    .000000000000000000

RESIDUAL 1-NORM/SOLUTION 1-NORM =   .465661286234845640-017


DO YOU WANT TO SEE THE RESIDUAL MATRIX (Y OR N)

>Y

RMFLG = Y

THE RESIDUAL MATRIX, BY ROWS:

.931322574615478516-009  .867361737988403547-018
.173472347597680709-017  -.237442714890844994-017


THE GAIN MATRIX IS:

20000.0000499999998    .333333332777777779-004
```

FIGURE A.2 Sample terminal session output listing (continued).

```
C     THIS IS AN INTERACTIVE MAIN DRIVER PROGRAM FOR THE SOFTWARE
C     PACKAGE RICPACK.  THIS DRIVER IS FOR SOLVING THE CONTINUOUS-
C     TIME OR DISCRETE-TIME GENERALIZED OPTIMAL REGULATOR PROBLEM.
C     THE PROGRAM PROMPTS FOR ALL NECESSARY INPUT.  FOR THE PROBLEM
C     SPECIFICATIONS AND DETAILS ON THE SOLUTION METHOD, SEE
C
C     REF.:  ARNOLD, W.F., "ON THE NUMERICAL SOLUTION OF
C     ALGEBRAIC MATRIX RICCATI EQUATIONS," PHD THESIS, USC,
C     DECEMBER 1983.
C
C     HISTORY:
C     THIS DRIVER WAS WRITTEN BY W.F. ARNOLD, NAVAL WEAPONS CENTER,
C     CODE 35104, CHINA LAKE, CA  93555, AS PART OF THE SOFTWARE
C     PACKAGE RICPACK, SEPTEMBER 1983.
C
C     SUBROUTINES CALLED:
C     BALANC, BALCOR, BCORBK, CMPRS, ELMHES, FBGAIN, HQR, LYPCND,
C     MADD, MLINEQ, MOUT, MQFWO, MSCALE, MSUB, MULB, NEWT, RESID,
C     RICSOL, RINV, SAVE, SEPEST, SEQUIV, TRNATB
C
C     *****FUNCTION SUBPROGRAMS:
      DOUBLE PRECISION D1NRM
      INTEGER IND(40),I,IBAL,INFO, IORD, J, L, L1, L2, L3, M, MAXIT,
     X    N, NN, NNPI, NNPJ, NNPM, NOUT, NPI, NPJ, NP1, NR, NRD, NRT
      CHARACTER CFLAG, EFLAG, KFLAG, NFLAG, QFLAG, RDFLG, RFLAG,
     X    RMFLG, RSFLG, SFLAG
      DOUBLE PRECISION A(20, 20), AS(20, 20), B(20, 20), C(20, 20),
     X    CQC(20, 20), E(20, 20), F(40, 40), G(40, 40), Q(20, 20),
     X    R(20, 20), RS(20, 20), S(20, 20), U(60, 60), WK(60, 20),
     X    Z(40, 40), ALFI(40), ALFR(40), BETA(60), CPERM(40),
     X    CSCALE(40), CA, CB, CLT1N, CL1N, COND, C1N, DG, DGI, DGN,
     X    RSD, RTOL, SEP, SR, TEMP, X1N
      LOGICAL TYPE
      DATA NR, NRD, NRT / 20, 40, 60 /
      RTOL = 1.0D + 06
      INFO = 0
      NOUT = 6
      TYPE = .TRUE.
      CFLAG = 'Y'
      WRITE (NOUT, 800) NR
C
C     READ IN PROBLEM PARAMETERS
C
      READ (5, *) N
      WRITE (NOUT, 802) N
      IF (N.GT.0) GOTO 10
      TYPE = .FALSE.
      N = - N
   10 CONTINUE
      IF (N.LE.NR .AND. N.NE.0) GOTO 20
```

```
      WRITE (NOUT, 804)
      STOP
   20 CONTINUE
      WRITE (NOUT, 806)
      READ (5, *) M, L
      WRITE (NOUT, 808) M, L
      IF (M.EQ.0) WRITE (NOUT, 810)
      IF (L.NE.0) GOTO 30
      WRITE (NOUT, 812)
      STOP
   30 CONTINUE
      WRITE (NOUT, 814)
      READ (5, *) IORD
      WRITE (NOUT, 816) IORD
      IF (M.EQ.0) IORD = - 1
      WRITE (NOUT, 818)
      READ (5, *) IBAL
      WRITE (NOUT, 820) IBAL
      IF (IBAL.NE.0 .AND. IBAL.NE.1) WRITE (NOUT, 822)
      WRITE (NOUT, 824)
      READ (5, 834) KFLAG
      WRITE (NOUT, 826) KFLAG
      IF (KFLAG.NE.'Y') GOTO 40
      WRITE (NOUT, 828)
      READ (5, *) DGI
      WRITE (NOUT, 830) DGI
   40 CONTINUE
      NN = 2 * N
      NNPM = NN + M
      IF (IBAL.EQ.1) GOTO 50
C
C     READ IN PROBLEM MODEL MATRICES
C
      WRITE (NOUT, 832)
      READ (5, 834) EFLAG
      WRITE (NOUT, 836) EFLAG
      IF (EFLAG.EQ.'Y') GOTO 60
   50 CONTINUE
      WRITE (NOUT, 838)
      GOTO 80
   60 CONTINUE
      WRITE (NOUT, 840) N, N
      DO 70 I = 1, N
          READ (5, *) (E(I, J), J = 1, N)
   70 CONTINUE
      WRITE (NOUT, 842)
      CALL MOUT(NOUT, NR, N, N, E)
   80 CONTINUE
      WRITE (NOUT, 844) N, N
      DO 90 I = 1, N
```

```
      READ (5, *) (A(I, J), J = 1, N)
 90 CONTINUE
    CALL SAVE(NR, NR, N, N, A, AS)
    WRITE (NOUT, 846)
    CALL MOUT(NOUT, NR, N, N, A)
    IF (M.EQ.0) GOTO 110
    WRITE (NOUT, 848) N, M
    DO 100 I = 1, N
      READ (5, *) (B(I, J), J = 1, M)
100 CONTINUE
    WRITE (NOUT, 850)
    CALL MOUT(NOUT, NR, N, M, B)
110 CONTINUE
    WRITE (NOUT, 852) L, N
    DO 120 I = 1, L
      READ (5, *) (C(I, J), J = 1, N)
120 CONTINUE
    WRITE (NOUT, 854)
    CALL MOUT(NOUT, NR, L, N, C)
    CALL SAVE(NR, NR, L, N, C, CQC)
    IF (M.EQ.0) GOTO 180
    WRITE (NOUT, 856)
    READ (5, 834) RFLAG
    WRITE (NOUT, 858) RFLAG
    IF (RFLAG.EQ.'Y') GOTO 130
    WRITE (NOUT, 860)
    GOTO 180
130 CONTINUE
    WRITE (NOUT, 862)
    READ (5, 834) RDFLG
    WRITE (NOUT, 864) RDFLG
    IF (RDFLG.NE.'Y') GOTO 160
    WRITE (NOUT, 866) M
    DO 150 J = 1, M
      DO 140 I = 1, M
        R(I, J) = 0.0D0
140   CONTINUE
150 CONTINUE
    READ (5, *) (R(I, I), I = 1, M)
    WRITE (NOUT, 868)
    CALL MOUT(NOUT, NR, M, M, R)
    GOTO 180
160 CONTINUE
    WRITE (NOUT, 870) M, M
    DO 170 I = 1, M
      READ (5, *) (R(I, J), J = 1, M)
170 CONTINUE
    WRITE (NOUT, 868)
    CALL MOUT(NOUT, NR, M, M, R)
180 CONTINUE
```

```
      IF (KFLAG.EQ.'Y') GOTO 230
      WRITE (NOUT, 872)
      READ (5, 834) QFLAG
      WRITE (NOUT, 874) QFLAG
      IF (QFLAG.EQ.'Y') GOTO 190
      WRITE (NOUT, 876)
      GOTO 210
  190 CONTINUE
      WRITE (NOUT, 878) L, L
      DO 200 I = 1, L
          READ (5, *) (Q(I, J), J = 1, L)
  200 CONTINUE
      WRITE (NOUT, 880)
      CALL MOUT(NOUT, NR, L, L, Q)
  210 CONTINUE
C
C     FORM THE MATRIX PRODUCT CT*Q*C AND STORE IN CQC
C
      CALL TRNATB(NR, NRD, L, N, C, G)
      IF (QFLAG.NE.'Y') GOTO 220
      CALL MULB(NR, NR, L, L, N, Q, CQC, CPERM)
  220 CONTINUE
      CALL MULB(NRD, NR, N, L, N, G, CQC, CPERM)
      GOTO 320
C
C     FORM ROBUSTNESS RECOVERY TERM
C
  230 CONTINUE
      ITER = 1
      WRITE (NOUT, 882)
      READ (5, 834) QFLAG
      WRITE (NOUT, 874) QFLAG
      IF (QFLAG.EQ.'Y') GOTO 240
      WRITE (NOUT, 884)
      GOTO 260
  240 CONTINUE
      WRITE (NOUT, 886) N, N
      DO 250 I = 1, N
          READ (5, *) (Q(I, J), J = 1, N)
  250 CONTINUE
      WRITE (NOUT, 880)
      CALL MOUT(NOUT, NR, N, N, Q)
  260 CONTINUE
      CALL TRNATB(NR, NRD, L, N, C, G)
      CALL MULB(NRD, NR, N, L, N, G, CQC, CPERM)
      CALL SAVE(NR, NRT, N, N, CQC, WK)
      CALL MSCALE(NR, N, N, DGI, CQC)
      DG = DGI
      IF (QFLAG.EQ.'Y') GOTO 280
      DO 270 I = 1, N
```

```
          CQC(I, I) = CQC(I, I) + 1.0D0
  270 CONTINUE
      GOTO 290
  280 CONTINUE
      CALL MADD(NR, NR, NR, N, N, CQC, Q, CQC)
  290 CONTINUE
      DO 310 I = 1, N
          DO 300 J = 1, N
              WK(J+N, I) = WK(J, I)
  300     CONTINUE
  310 CONTINUE
  320 CONTINUE
      IF (M.EQ.0) GOTO 350
      WRITE (NOUT, 888)
      READ (5, 834) SFLAG
      WRITE (NOUT, 890) SFLAG
      IF (SFLAG.EQ.'Y') GOTO 330
      WRITE (NOUT, 892)
      GOTO 350
  330 CONTINUE
      WRITE (NOUT, 894) N, M
      DO 340 I = 1, N
          READ (5, *) (S(I, J), J = 1, M)
  340 CONTINUE
      WRITE (NOUT, 896)
      CALL MOUT(NOUT, NR, N, M, S)
  350 CONTINUE
C
C     CALCULATE CO-ORDINATE BALANCING TRANSFORMATION IF REQUESTED
C
      IF (IBAL.NE.1) GOTO 360
      CALL BALCOR(NR, NRD, L, M, N, A, B, C, CQC, S, E, Q, Z, ALFI,
     X    ALFR, BETA, IND, INFO, SFLAG, TYPE)
      IF (INFO.EQ.0) GOTO 360
      WRITE (NOUT, 898)
      IF (INFO.EQ.1) WRITE (NOUT, 900)
      IF (INFO.EQ.-1) WRITE (NOUT, 902)
      IF (INFO.EQ.-2) WRITE (NOUT, 904)
      WRITE (NOUT, 906)
      READ (5, 834) RSFLG
      IF (RSFLG.NE.'Y') STOP
      IBAL = 0
  360 CONTINUE
C
C     TAKE INVERSE OF R MATRIX AND STORE IN Q
C
      IF (RFLAG.NE.'Y') GOTO 420
      IF (RDFLG.NE.'Y') GOTO 380
      DO 370 I = 1, M
          Q(I, I) = 1.0D0 / R(I, I)
```

```
  370 CONTINUE
      GOTO 420
  380 CONTINUE
      CALL SAVE(NR, NRD, M, M, R, G)
      DO 400 J = 1, M
          DO 390 I = 1, M
              Q(I, J) = 0.0D0
  390     CONTINUE
          Q(J, J) = 1.0D0
  400 CONTINUE
      CALL MLINEQ(NRD, NR, M, M, G, Q, COND, IND, CPERM)
      WRITE (NOUT, 908) COND
      IF (COND.LT.RTOL) GOTO 410
      WRITE (NOUT, 910)
      READ (5, 834) CFLAG
      IF (CFLAG.NE.'Y') GOTO 430
  410 CONTINUE
      WRITE (NOUT, 912)
  420 CONTINUE
C
C     SET UP MATRIX PENCIL FOR CONTINUOUS-TIME PROBLEM USING
C     R INVERSE
C
      IF (SFLAG.EQ.'Y') CALL
     X    SEQUIV(NR, NRD, M, N, A, B, CQC, S, Q, F, G, RDFLG, RFLAG)
      CALL RINV(NR, NRD, N, NN, M, E, A, B, CQC, Q, G, F, RS, CPERM,
     X    RDFLG, RFLAG, EFLAG, IBAL, TYPE)
      GOTO 440
  430 CONTINUE
C
C     SET UP MATRIX PENCIL FOR CONTINUOUS-TIME PROBLEM USING
C     COMPRESSION TECHNIQUE WHEN R IS ILL-CONDITIONED WITH
C     RESPECT TO INVERSION
C
      WRITE (NOUT, 914)
      IF (.NOT.TYPE) CFLAG = 'Y'
      KFLAG = 'N'
      CALL CMPRS(NR, NRD, NRT, N, NN, NNPM, M, E, A, B, CQC, R, S, G,
     X    F, U, WK, CPERM, CSCALE, BETA, EFLAG, SFLAG, IBAL, INFO)
      IF (INFO.NE.0) GOTO 720
  440 CONTINUE
      IF (TYPE) GOTO 470
C
C     CONVERT PENCIL TO DISCRETE-TIME PENCIL IF DISCRETE PROBLEM
C
      NP1 = N + 1
      DO 460 J = NP1, NN
          DO 450 I = 1, NN
              TEMP = G(I, J)
              G(I, J) = F(I, J)
```

```
                F(I, J) = - TEMP
  450       CONTINUE
  460   CONTINUE
  470   CONTINUE
        IF (IBAL.EQ.1) CALL BCORBK(NR, NRD, M, N, A, B, CQC, S, Z)
C
C       SAVE PENCIL IF ITERATING FOR ROBUSTNESS RECOVERY
C
        IF (KFLAG.NE.'Y') GOTO 500
        CALL SAVE(NRD, NRT, NN, NN, G, U)
        DO 490 I = 1, N
            NPI = N + I
            NNPI = NN + I
            DO 480 J = 1, N
                NPJ = N + J
                NNPJ = NN + J
                U(NNPJ, I) = F(J, I)
                U(NNPJ, NPI) = F(J, NPI)
                U(J, NNPI) = F(NPJ, I)
                U(NPJ, NNPI) = F(NPJ, NPI)
  480       CONTINUE
  490   CONTINUE
  500   CONTINUE
C
C       COMPUTE THE RICCATI SOLUTION
C
        CALL RICSOL(NR, NRD, NN, N, G, F, E, Z, ALFR, ALFI, BETA,
       X    CPERM, CSCALE, IND, IORD, IBAL, TYPE, EFLAG)
        IF (IND(1).NE.0) GOTO 690
        INFO = IND(2)
        IF (CPERM(1).EQ.1.0D + 20) CALL SAVE(NRD, NRD, N, N, Z, F)
        IF (IBAL.NE.1) GOTO 530
C
C       ESTIMATE CONDITION OF CO-ORDINATE BALANCING TRANSFORMATION
C       WITH RESPECT TO INVERSION
C
        DO 520 J = 1, N
            DO 510 I = 1, N
                G(I, J) = 0.0D0
  510       CONTINUE
            G(J, J) = 1.0D0
  520   CONTINUE
        CALL SAVE(NR, NRD, N, N, E, Z)
        CALL MLINEQ(NRD, NRD, N, N, Z, G, COND, IND, CSCALE)
        WRITE (NOUT, 916) COND
  530   CONTINUE
C
C       OUTPUT THE SOLUTION
C
        WRITE (NOUT, 918)
```

```
      IF (.NOT.TYPE) GOTO 560
      WRITE (NOUT, 920)
      DO 550 I = 1, NN
          IF (ALFR(I).GE.0.0D0) GOTO 550
          IF (BETA(I).NE.0.0D0) GOTO 540
          WRITE (NOUT, 922)
          GOTO 550
  540     CONTINUE
          ALFR(I) = ALFR(I) / BETA(I)
          ALFI(I) = ALFI(I) / BETA(I)
          WRITE (NOUT, *) ALFR(I), ALFI(I)
  550 CONTINUE
      GOTO 580
  560 CONTINUE
      WRITE (NOUT, 924)
      DO 570 I = 1, NN
          IF (BETA(I).EQ.0.0D0) GOTO 570
          IF (DABS(ALFR(I)).GE.BETA(I)) GOTO 570
          ALFR(I) = ALFR(I) / BETA(I)
          ALFI(I) = ALFI(I) / BETA(I)
          WRITE (NOUT, *) ALFR(I), ALFI(I)
  570 CONTINUE
  580 CONTINUE
      IF (INFO.NE.0) GOTO 700
      IF (CPERM(1).EQ.1.0D + 20) GOTO 710
      IF (M.EQ.0) WRITE (NOUT, 926)
      IF (IORD.EQ.-1) WRITE (NOUT, 928)
      IF (IORD.EQ.0) WRITE (NOUT, 930)
      IF (IORD.EQ.1) WRITE (NOUT, 932)
      CALL MOUT(NOUT, NR, N, N, F)
C
C     COMPUTE CONDITION ESTIMATES
C
      X1N = D1NRM(NRD, N, N, F)
      CALL SAVE(NR, NR, N, N, CQC, C)
      IF (SFLAG.EQ.'Y') CALL
     X    SEQUIV(NR, NRD, M, N, AS, B, C, S, Q, Z, G, RDFLG, RFLAG)
      C1N = D1NRM(NR, N, N, C)
      CALL FBGAIN(NR, NRD, NRD, N, M, A, B, E, R, Q, S, F, Z, G,
     X    CSCALE, IND, EFLAG, RDFLG, RFLAG, SFLAG, TYPE)
      WRITE (NOUT, 934) X1N, C1N, CPERM(1)
      CALL MULB(NR, NRD, N, M, N, B, Z, CSCALE)
      CALL MSUB(NR, NRD, NRD, N, N, A, Z, Z)
      IF (TYPE) GOTO 600
      CL1N = D1NRM(NRD, N, N, Z)
      CALL TRNATA(NRD, N, Z)
      CLT1N = D1NRM(NRD, N, N, Z)
      CALL BALANC(NRD, N, Z, L1, L2, CSCALE)
      CALL ELMHES(NRD, N, L1, L2, Z, IND)
      CALL HQR(NRD, N, L1, L2, Z, ALFR, ALFI, L3)
```

```
      IF (L3.NE.0) WRITE (NOUT, 936)
      SR = 0.0D0
      DO 590 I = 1, N
          SR = DMAX1(SR, DSQRT(ALFR(I)*ALFR(I) + ALFI(I)*ALFI(I)))
  590 CONTINUE
      TEMP = 1.0D0 - CL1N * CLT1N
      IF (TEMP.LE.0.0D0) TEMP = 1.0D0 - SR * SR
      CA = DFLOAT(N) * (C1N / X1N + 2.0D0 * CL1N * CLT1N) / TEMP
      WRITE (NOUT, 938) CA, SR, CL1N, CLT1N
      GOTO 620
  600 CONTINUE
      L1 = 0
      L2 = 1
      CALL LYPCND(NRD, NRD, N, Z, F, G, ALFR, ALFI, BETA, L1, L2)
      CALL SEPEST(NRD, N, Z, G, SEP, L1)
      IF (L1.NE.0) WRITE (NOUT, 940)
      TEMP = DABS(ALFR(1))
      DO 610 I = 2, N
          TEMP = DMIN1(TEMP, DABS(ALFR(I)))
  610 CONTINUE
      CB = (C1N / X1N + 2.0D0 * D1NRM(NR, N, N, AS) + X1N *
     X    D1NRM(NR, N, N, RS)) / SEP
      CA = C1N / (X1N * SEP)
      TEMP = 2.0D0 * TEMP
      WRITE (NOUT, 942) CB, CA, TEMP, SEP
  620 CONTINUE
C
C     SET UP FOR ANOTHER ROBUSTNESS RECOVERY ITERATION
C
      IF (KFLAG.NE.'Y') GOTO 680
      WRITE (NOUT, 944)
      READ (5, 834) KFLAG
      WRITE (NOUT, 826) KFLAG
      IF (KFLAG.NE.'Y') GOTO 650
      WRITE (NOUT, 946)
      READ (5, *) DGN
      WRITE (NOUT, 830) DGN
      IF (IBAL.EQ.1 .AND. ITER.EQ.1). CALL
     X    MQFWO(NRT, NR, N, WK, E, CPERM)
      ITER = ITER + 1
      DG = DG - DGN
      DO 640 I = 1, N
          NPI = N + I
          NNPI = NN + I
          DO 630 J = 1, N
              NPJ = N + J
              NNPJ = NN + J
              U(NPJ, I) = U(NPJ, I) + DG * WK(J, I)
              F(J, I) = U(NNPJ, I)
              F(J, NPI) = U(NNPJ, NPI)
```

```
                F(NPJ, I) = U(J, NNPI)
                F(NPJ, NPI) = U(NPJ, NNPI)
  630       CONTINUE
  640   CONTINUE
        CALL SAVE(NRT, NRD, NN, NN, U, G)
        DG = DGN
        GOTO 500
  650   CONTINUE
        DG = DG - DGI
        DO 670 I = 1, N
            DO 660 J = 1, N
                CQC(J, I) = CQC(J, I) + DG * WK(J+N, I)
  660       CONTINUE
  670   CONTINUE
  680   CONTINUE
        IF (CFLAG.NE.'Y') STOP
C
C       COMPUTE THE RESIDUAL
C
        WRITE (NOUT, 948)
        READ (5, 834) RSFLG
        WRITE (NOUT, 950) RSFLG
        IF (RSFLG.NE.'Y') GOTO 730
        CALL RESID(NR, NR, NRD, NRD, N, M, E, A, B, CQC, R, S, Q, RS,
       X    F, G, Z, CPERM, IND, RTOL, EFLAG, RFLAG, RDFLG, SFLAG,
       X    RSD, TYPE, NOUT)
        WRITE (NOUT, 952) RSD
        WRITE (NOUT, 954)
        READ (5, 834) RMFLG
        WRITE (NOUT, 956) RMFLG
        IF (RMFLG.NE.'Y') GOTO 730
        WRITE (NOUT, 958)
        CALL MOUT(NOUT, NR, N, N, RS)
        GOTO 730
  690   CONTINUE
        WRITE (NOUT, 960)
        STOP
  700   CONTINUE
        WRITE (NOUT, 962)
        STOP
  710   CONTINUE
        WRITE (NOUT, 964)
        CALL MOUT(NOUT, NRD, N, N, F)
        STOP
  720   CONTINUE
        WRITE (NOUT, 966)
        STOP
  730   CONTINUE
        IF (IORD.NE.-1) STOP
C
```

```
C     APPLY NEWTON'S METHOD FOR ITERATIVE IMPROVEMENT
C
      WRITE (NOUT, 968)
      READ (5, 834) NFLAG
      WRITE (NOUT, 970) NFLAG
      IF (NFLAG.NE.'Y') GOTO 750
      WRITE (NOUT, 972)
      READ (5, *) MAXIT
      WRITE (NOUT, 974) MAXIT
      CALL SAVE(NRD, NRT, N, N, F, U)
      CALL NEWT(NR, NR, NRD, NRT, N, M, E, A, B, CQC, R, S, Q, RS,
     X    U, G, F, Z, ALFR, ALFI, BETA, IND, RTOL, EFLAG, RFLAG,
     X    RDFLG, SFLAG, SEP, TYPE, MAXIT, INFO, NOUT)
      IF (INFO.NE.0) GOTO 760
      WRITE (NOUT, 976) MAXIT
      WRITE (NOUT, 978) IND(1)
      CALL RESID(NR, NR, NRD, NRT, N, M, E, A, B, CQC, R, S, Q, RS,
     X    U, G, Z, CPERM, IND, RTOL, EFLAG, RFLAG, RDFLG, SFLAG,
     X    RSD, TYPE, NOUT)
      WRITE (NOUT, 980)
      CALL MOUT(NOUT, NRT, N, N, U)
      WRITE (NOUT, 918)
      DO 740 I = 1, N
          WRITE (NOUT, *) ALFR(I), ALFI(I)
  740 CONTINUE
      WRITE (NOUT, 952) RSD
      CALL SAVE(NRT, NRD, N, N, U, F)
      WRITE (NOUT, 954)
      READ (5, 834) RMFLG
      WRITE (NOUT, 956) RMFLG
      IF (RMFLG.NE.'Y') GOTO 750
      WRITE (NOUT, 958)
      CALL MOUT(NOUT, NR, N, N, RS)
  750 CONTINUE
C
C     CALCULATE OPTIMAL FEEDBACK GAIN MATRIX
C
      CALL FBGAIN(NR, NRD, NRD, N, M, A, B, E, R, Q, S, F, Z, G,
     X    CPERM, IND, EFLAG, RDFLG, RFLAG, SFLAG, TYPE)
      IF (CPERM(1).GT.RTOL) WRITE (NOUT, 984) CPERM(1)
      WRITE (NOUT, 982)
      CALL MOUT(NOUT, NRD, M, N, Z)
      STOP
  760 CONTINUE
      IF (INFO.EQ.-3) GOTO 770
      WRITE (NOUT, 986) INFO
      CALL MOUT(NOUT, NRT, N, N, U)
      WRITE (NOUT, 988) BETA(1)
      CALL RESID(NR, NR, NRD, NRT, N, M, E, A, B, CQC, R, S, Q, RS,
     X    U, G, Z, CPERM, IND, RTOL, EFLAG, RFLAG, RDFLG, SFLAG,
```

```
     X    RSD, TYPE, NOUT)
          GOTO 740
  770 CONTINUE
          WRITE (NOUT, 990)
          STOP
  800 FORMAT(/' GENERALIZED ALGEBRAIC MATRIX RICCATI EQUATION SOLVER',
     X    //,' ENTER SYSTEM ORDER "N":  + FOR CONTINUOUS TIME PROBLE',
     X    'MS, - FOR DISCRETE', /, ' MAXIMUM ORDER =', I3, /)
  802 FORMAT (/' N =', I3/)
  804 FORMAT (//' ORDER EXCEEDS MAXIMUM *** EXECUTION TERMINATED', /)
  806 FORMAT (//' ENTER NUMBER OF SYSTEM INPUTS "M"', /,
     X    ' AND NUMBER OF SYSTEM OUTPUTS "L"'/)
  808 FORMAT (/' M =', I3, 5X, 'L =', I3/)
  810 FORMAT (/' SYSTEM HAS NO INPUTS, RICCATI EQUATION DEGENERATES',
     X    ' TO A LYAPUNOV EQUATION'/)
  812 FORMAT (//' SYSTEM HAS NO OUTPUT *** EXECUTION TERMINATED'/)
  814 FORMAT (/' ENTER FLAG FOR DESIRED SOLUTION:', /,
     X    ' -1 FOR STABILIZING SOLUTION',/,' 0 FOR ANY SOLUTION', /,
     X    ' +1 FOR DESTABILIZING SOLUTION'/)
  816 FORMAT (/' IORD = ', I2/)
  818 FORMAT (/' ENTER BALANCING FLAG:  0 FOR WARD BALANCING', /,
     X    '                          1 FOR CO-ORDINATE BALANCING'/,
     X    '                          2 FOR NO BALANCING'/)
  820 FORMAT (/' IBAL =', I2/)
  822 FORMAT (/' NO BALANCING ATTEMPTED'/)
  824 FORMAT
     X    (/' DO YOU WISH TO ITERATE FOR ROBUSTNESS RECOVERY',
     X    ' (Y OR N)'/)
  826 FORMAT (/' KFLAG = ', A1/)
  828 FORMAT (/' ENTER INITIAL GAMMA PARAMETER'/)
  830 FORMAT (/' DG =', D26.18/)
  832 FORMAT (/' DO YOU WISH TO ENTER AN "E" MATRIX (Y OR N)', /,
     X    ' DEFAULT IS E = IDENTITY MATRIX'/)
  834 FORMAT (1A1)
  836 FORMAT (/' EFLAG = ', A1/)
  838 FORMAT (/' USING DEFAULT:  "E" = IDENTITY'/)
  840 FORMAT (/' ENTER THE ', I3, ' X ', I3, ' MATRIX "E" BY ROWS'/)
  842 FORMAT (/' THE "E" MATRIX IS:'/)
  844 FORMAT
     X    (/' ENTER THE ',I3,' X ',I3,' SYSTEM MATRIX "A" BY ROWS'/)
  846 FORMAT (/' THE "A" MATRIX IS:'/)
  848 FORMAT
     X    (/' ENTER THE ', I3,' X ',I3,' INPUT MATRIX "B" BY ROWS'/)
  850 FORMAT (/' THE "B" MATRIX IS:'/)
  852 FORMAT
     X    (/' ENTER THE ', I3,' X ',I3,' OUTPUT MATRIX "C" BY ROWS'/)
  854 FORMAT (/' THE "C" MATRIX IS:'/)
  856 FORMAT(/' DO YOU WISH TO ENTER A CONTROL WEIGHTING MATRIX "R"',
     X    ' (Y OR N)', /, ' DEFAULT IS R = IDENTITY MATRIX'/)
  858 FORMAT (/' RFLAG = ', A1/)
```

```
860 FORMAT (/' USING DEFAULT:  "R" = IDENTITY'/)
862 FORMAT(/' DO YOU WISH TO ENTER A DIAGONAL "R" MATRIX(Y OR N)'/)
864 FORMAT (/' RDFLG = ', A1/)
866 FORMAT (/' ENTER THE ', I3, ' DIAGONAL ELEMENTS AS A ROW'/)
868 FORMAT (/' THE "R" MATRIX IS:'/)
870 FORMAT (/' ENTER THE ', I3, ' X ',I3,' INPUT WEIGHTING MATRIX',
   X    ' "R" BY ROWS'/)
872 FORMAT(/' DO YOU WISH TO ENTER AN OUTPUT WEIGHTING MATRIX "Q"',
   X    ' (Y OR N)', /, ' DEFAULT IS Q = IDENTITY MATRIX'/)
874 FORMAT (/' QFLAG = ', A1/)
876 FORMAT (/' USING DEFAULT:  "Q" = IDENTITY'/)
878 FORMAT (/' ENTER THE ', I3, ' X ', I3,
   X    ' OUTPUT WEIGHTING MATRIX "Q"', ' BY ROWS'/)
880 FORMAT (/' THE "Q" MATRIX IS:'/)
882 FORMAT (/' DO YOU WISH TO ENTER A STATE WEIGHTING MATRIX "Q"',
   X    ' (Y OR N)', /, ' DEFAULT IS Q = IDENTITY MATRIX'/)
884 FORMAT (/' USING DEFAULT:  "Q" = IDENTITY'/)
886 FORMAT (/' ENTER THE', I3, ' X ', I3,
   X    ' STATE WEIGHTING MATRIX "Q"', ' BY ROWS'/)
888 FORMAT (/' DO YOU WISH TO ENTER A STATE/INPUT CROSS-WEIGHTING',
   X    ' MATRIX "S" (Y OR N)', /, ' DEFAULT IS S = ZERO MATRIX'/)
890 FORMAT (/' SFLAG = ', A1/)
892 FORMAT (/' USING DEFAULT:  "S" = ZERO MATRIX'/)
894 FORMAT (/' ENTER THE ', I3, ' X ', I3,
   X    ' CROSS-WEIGHTING MATRIX "S"', ' BY ROWS'/)
896 FORMAT (/' THE "S" MATRIX IS:'/)
898 FORMAT (/' CO-ORDINATE BALANCING UNSUCCESSFUL BECAUSE:')
900 FORMAT (' CAN NOT COMPUTE SOLUTION TO LYAPUNOV EQUATION'/)
902 FORMAT (' CONTROLLABILITY GRAMMIAN NOT NUMERICALLY P.D.'/)
904 FORMAT (' CONVERGENCE FAILURE IN EIGENVALUE ROUTINE'/)
906 FORMAT(/' DO YOU WISH TO CONTINUE WITH WARD BALANCING(Y OR N)'/)
908 FORMAT
   X    (/' INVERSION CONDITION ESTIMATE FOR "R" MATRIX =',D26.18,/)
910 FORMAT (/' DO YOU WISH TO CONTINUE USING "R" INVERSE (Y OR N)'/)
912 FORMAT (/' PROCEEDING WITH SOLUTION USING "R" INVERSE'/)
914 FORMAT (/' PROCEEDING WITH COMPRESSION TECHNIQUE'/)
916 FORMAT (/' A CONDITION ESTIMATE FOR THE CO-ORDINATE BALANCING',
   X    ' TRANSFORMATION IS:', /, D26.18/)
918 FORMAT (/
   X    ' THE CLOSED LOOP EIGENVALUES FOR THE STABILIZING RICCATI',
   X    ' SOLUTION ARE:'/)
920 FORMAT (' (FOR CONTINUOUS TIME)'/)
922 FORMAT (/' INFINITE EIGENVALUE'/)
924 FORMAT (' (FOR DISCRETE TIME)'/)
926 FORMAT (/' THE LYAPUNOV SOLUTION IS:'/)
928 FORMAT (/' THE STABILIZING RICCATI SOLUTION IS:'/)
930 FORMAT (/' A RICCATI SOLUTION IS:'/)
932 FORMAT (/' THE DESTABILIZING RICCATI SOLUTION IS'/)
934 FORMAT
   X    (/' X1N =',D15.8,5X,'C1N =',D15.8,5X, 'K(Z11) =', D15.8/)
```

```
936 FORMAT (/'ERROR IN HQR '/)
938 FORMAT (/' KA(X) =', D15.8, 10X, 'SR =', D15.8, /, ' CL1N =',
   X    D15.8, 10X, 'CLT1N =', D15.8/)
940 FORMAT (/' ERROR IN SEPEST'/)
942 FORMAT (/' KB(X) =', D15.8,10X,'KA(X) =',D15.8,/,' SEPEST =',
   X    D15.8, 10X, 'SEP =', D15.8, /)
944 FORMAT (/' DO YOU WISH TO ENTER ANOTHER GAMMA (Y OR N)'/)
946 FORMAT (/' ENTER NEW VALUE FOR GAMMA'/)
948 FORMAT (/' DO YOU WISH A RESIDUAL CALCULATION (Y OR N)'/)
950 FORMAT (/' RSFLG = ', A1/)
952 FORMAT (/' RESIDUAL 1-NORM/SOLUTION 1-NORM =', D26.18, /)
954 FORMAT (/' DO YOU WANT TO SEE THE RESIDUAL MATRIX (Y OR N)'/)
956 FORMAT (/' RMFLG = ', A1/)
958 FORMAT (/' THE RESIDUAL MATRIX, BY ROWS:'/)
960 FORMAT (/' MORE THAT 50 ITERATIONS REQUIRED BY QZITW', /,
   X    ' *** EXECUTION TERMINATED'/)
962 FORMAT (/' CONVERGENCE FAILURE IN ORDER ROUTINE', /,
   X    ' *** EXECUTION TERMINATED'/)
964 FORMAT(/' SCHUR VECTOR MATRIX SINGULAR TO WORKING PRECISION',/,
   X    ' THEREFORE, SOLUTION BY THIS TECHNIQUE NOT POSSIBLE', /,
   X    ' THE SCHUR VECTOR MATRIX IS:'/)
966 FORMAT (/' COMPRESSION FAILURE *** EXECUTION TERMINATED'/)
968 FORMAT (/' DO YOU WANT TO TRY ITERATIVE IMPROVEMENT (Y OR N)'/)
970 FORMAT (/' NFLAG = ', A1/)
972 FORMAT (/' ENTER MAXIMUM NUMBER OF NEWTON ITERATIONS'/)
974 FORMAT (/' MAX. NO. OF ITERATIONS =', I4/)
976 FORMAT
   X    (/' CONVERGED AFTER',I3,'  ITERATIONS OF NEWTONS METHOD'/)
978 FORMAT (' CONVERGENCE CRITERIA', I2/)
980 FORMAT (/' THE REFINED STABILIZING SOLUTION IS:'/)
982 FORMAT (/' THE GAIN MATRIX IS:'/)
984 FORMAT (//' R + GT*X*G ILL-CONDITIONED WRT INVERSION', /,
   X    ' CONDITION NUMBER =', D26.18, //)
986 FORMAT (/' NEWTON ITERATION FAILED, INFO = ', I3/,
   X    ' THE SOLUTION AT THE LAST ITERATION IS:'/)
988 FORMAT (/' THE 1-NORM OF THE ERROR IS:', D26.18/)
990 FORMAT(/' SORRY, PROGRAM NOT ABLE TO PERFORM NEWTON ITERATION',
   X    /, ' FOR ARBITRARY E-MATRIX AT THIS TIME!!!'/)
    END
```

```
      SUBROUTINE BALCOR (NR,NRZ,L,M,N,A,B,C,CQC,S,T,WK,Z,WK1,WK2,
     X                   WK3,IPVT,ISTAB,SFLAG,TYPE)
C
C     *****PARAMETERS:
      INTEGER NR,NRZ,L,M,N,IPVT(N),ISTAB
      CHARACTER SFLAG
      DOUBLE PRECISION A(NR,N),B(NR,M),C(NR,N),CQC(NR,N),S(NR,M),
     X    T(NR,N),WK(NR,N),Z(NRZ,N),WK1(N),WK2(N),WK3(N)
      LOGICAL TYPE
C
C     *****LOCAL VARIABLES:
      INTEGER I,J,NPI,NPJ
C
C     *****FORTRAN FUNCTIONS:
C     NONE.
C
C     *****SUBROUTINES CALLED:
C     BCORBK, BLCRDC, BLCRDD, MMUL, MQFWO, MULB, TRNATB
C
C     ------------------------------------------------------------
C
C     *****PURPOSE:
C     GIVEN THE MODEL MATRICES A, B AND C FOR A FIRST ORDER LINEAR
C     MODEL IN STATE SPACE FORM, THIS SUBROUTINE CALCULATES A
C     BALANCING TRANSFORMATION, T, SUCH THAT IF T WAS APPLIED TO THE
C     MODEL AS A CHANGE OF COORDINATES, THE REACHABILITY AND
C     OBSERVABILITY GRAMMIANS WOULD BE EQUAL AND DIAGONAL.  HOWEVER,
C     T IS ACTUALLY APPLIED TO THE MODEL MATRICES IN A SPECIAL WAY
C     FOR SOLUTION OF THE OPTIMAL REGULATOR PROBLEM BY RICPACK.
C
C     REF.:  ARNOLD, W.F., "ON THE NUMERICAL SOLUTION OF
C     ALGEBRAIC MATRIX RICCATI EQUATIONS," PHD THESIS, USC,
C     DECEMBER 1983.
C
C     *****PARAMETER DESCRIPTION:
C
C     ON INPUT:
C
C       NR      INTEGER
C               ROW DIMENSION OF THE ARRAYS CONTAINING THE A, B,
C               C, CQC, S, T AND WK MATRICES AS DECLARED IN THE
C               MAIN CALLING PROGRAM DIMENSION STATEMENT;
C
C       NRZ     INTEGER
C               ROW DIMENSION OF THE ARRAY CONTAINING THE Z MATRIX
C               AS DECLARED IN THE MAIN CALLING PROGRAM DIMENSION
C               STATEMENT;
C
C       L       INTEGER
```

```
C                ROW DIMENSION OF THE C MATRIX;
C
C        M       INTEGER
C                COLUMN DIMENSION OF THE B AND S MATRICES;
C
C        N       INTEGER
C                ORDER OF THE SQUARE MATRICES A, CQC AND T
C                COLUMN DIMENSION OF THE C MATRIX;
C
C        A       REAL(NR,N)
C                MODEL SYSTEM MATRIX, ALTERED BY THIS ROUTINE;
C
C        B       REAL(NR,M)
C                MODEL INPUT MATRIX;
C
C        C       REAL(NR,N)
C                MODEL OUTPUT MATRIX, ALTERED BY THIS ROUTINE;
C
C        CQC     REAL(NR,N)
C                MATRIX PRODUCT CT*Q*C WHERE T DENOTES MATRIX
C                TRANSPOSE, ALTERED BY THIS ROUTINE;
C
C        S       REAL(NR,M)
C                STATE - INPUT CROSS-WEIGHTING MATRIX, ALTERED BY
C                THIS ROUTINE;
C
C        WK      REAL(NR,N)
C                SCRATCH ARRAY OF SIZE AT LEAST N BY N;
C
C        WK1,WK2,WK3
C                REAL(N)
C                WORKING VECTORS OF SIZE AT LEAST N;
C
C        IPVT    INTEGER(N)
C                WORKING VECTOR OF SIZE AT LEAST N;
C
C        SFLAG   CHARACTER
C                FLAG SET TO 'Y' IF S.IS OTHER THAN THE ZERO MATRIX;
C
C        TYPE    LOGICAL
C                = .TRUE.  FOR A CONTINUOUS-TIME MODEL
C                = .FALSE. FOR A DISCRETE-TIME MODEL.
C
C     ON OUTPUT:
C
C        A       CONTAINS THE MATRIX PRODUCT F*T;
C
C        CQC     CONTAINS THE MATRIX PRODUCT (C*T)-TRANSPOSE*Q*C*T;
C
```

```
C       S       CONTAINS THE MATRIX PRODUCT T-TRANSPOSE*S;
C
C       T       REAL(NR,N)
C               CONTAINS THE BALANCING TRANSFORMATION, T;
C
C       Z       REAL(NRZ,N)
C               2N BY (N+M) MATRIX WITH THE ORIGINAL A MATRIX STORED
C               IN THE UPPER LEFT N BY N BLOCK, CQC IN THE LOWER LEFT
C               N BY N BLOCK, S IN THE UPPER RIGHT N BY M BLOCK AND
C               B IN THE LOWER RIGHT N BY M BLOCK;
C
C       ISTAB   INTEGER
C               ERROR FLAG WITH THE FOLLOWING MEANINGS
C               = ZERO, NORMAL RETURN
C               = NONZERO, A BALANCING TRANSFORMATION COULD NOT BE
C                 CALCULATED AND THE A, CQC AND S MATRICES ARE
C                 RETURNED UNALTERED.
C
C     *****ALGORITHM NOTES:
C     NONE.
C
C     *****HISTORY:
C     THIS SUBROUTINE WAS WRITTEN BY W.F. ARNOLD, NAVAL WEAPONS
C     CENTER, CODE 35104, CHINA LAKE, CA  93555, AS PART OF THE
C     SOFTWARE PACKAGE RICPACK, SEPTEMBER 1983.
C
C     ------------------------------------------------------------
C
```

```
      SUBROUTINE RICSOL(NR,NRD,NN,N,G,F,E,Z,ALFR,ALFI,BETA,CPERM,
     X                  CSCALE,IND,IORD,IBAL,TYPE,EFLAG)
C
C     *****PARAMETERS:
      INTEGER NR,NRD,NN,N,IND(NN),IORD,IBAL
      CHARACTER EFLAG
      DOUBLE PRECISION G(NRD,NN),F(NRD,NN),E(NR,N),Z(NRD,NN),
     X          ALFR(NN),ALFI(NN),BETA(NN),CPERM(NN),CSCALE(NN)
      LOGICAL TYPE
C
C     *****LOCAL VARIABLES:
      INTEGER I,IERR,IFAIL,IGH,J,LOW,NPJ
      DOUBLE PRECISION COND,EPS,EPS1
      LOGICAL MATZ
C
C     *****FORTRAN FUNCTIONS:
C     NONE.
C
C     *****SUBROUTINES CALLED:
C     BALGBK, BALGEN, MLINEQ, MULB, ORDER, QZHESW, QZITW, QZVAL
C
C     ------------------------------------------------------------
C
C     *****PURPOSE:
C     GIVEN THE 2N BY 2N MATRIX PENCIL
C
C              LAMBDA * F  -  G
C
C     THIS SUBROUTINE FINDS THE ORTHOGONAL MATRIX Z SUCH THAT
C
C              Q * (LAMBDA * F  -  G) * Z
C
C     IS IN GENERALIZED ORDERED REAL SCHUR FORM.  FURTHERMORE, THE
C     UPPER LEFT N BY N BLOCK OF THE TRANSFORMED PENCIL CONTAINS
C     THE EIGENVALUES SPECIFIED BY THE PARAMETER IORD.  THE
C     SUBROUTINE THEN SOLVES THE LINEAR SYSTEM
C
C              X * E * Z11  =  Z21
C
C     FOR X, WHERE Z11 AND Z21 ARE THE UPPER AND LOWER LEFT N BY N
C     BLOCKS OF Z.
C
C     REF.:  ARNOLD, W.F., "ON THE NUMERICAL SOLUTION OF
C     ALGEBRAIC MATRIX RICCATI EQUATIONS," PHD THESIS, USC,
C     DECEMBER 1983.
C
C     *****PARAMETER DESCRIPTION:
C
C     ON INPUT:
```

```
C
C       NR      INTEGER
C               ROW DIMENSION OF THE ARRAY CONTAINING THE MATRIX E
C               AS DECLARED IN THE MAIN CALLING PROGRAM DIMENSION
C               STATEMENT;
C
C       NRD     INTEGER
C               ROW DIMENSION OF THE ARRAYS CONTAINING THE MATRICES
C               G, F AND Z AS DECLARED IN THE MAIN CALLING PROGRAM
C               DIMENSION STATEMENT;
C
C       NN      INTEGER
C               ORDER OF THE SQUARE MATRICES G AND F;
C
C       N       INTEGER
C               ORDER OF THE SQUARE MATRIX E;
C
C       G       REAL(NRD,NN)
C               PENCIL MATRIX CORRESPONDING TO THE GENERALIZED RICCATI
C               PROBLEM, WHOSE CONTENTS ARE ALTERED BY THIS ROUTINE;
C
C       F       REAL(NRD,NN)
C               PENCIL MATRIX CORRESPONDING TO THE GENERALIZED RICCATI
C               PROBLEM, WHOSE CONTENTS ARE ALTERED BY THIS ROUTINE;
C
C       E       REAL(NR,N)
C               DESCRIPTOR MATRIX OR CO-ORDINATE BALANCING MATRIX
C               AS SPECIFIED BY THE PARAMETER IBAL;
C
C       CPERM   REAL(NN)
C               WORKING VECTOR OF SIZE AT LEAST NN;
C
C       CSCALE  REAL(NN)
C               WORKING VECTOR OF SIZE AT LEAST NN;
C
C       IND     INTEGER(NN)
C               WORKING VECTOR OF SIZE AT LEAST NN;
C
C       IORD    INTEGER
C               PARAMETER SPECIFYING THE SPECTRUM OF THE UPPER LEFT N
C               BY N BLOCK OF THE ORDERED REAL SCHUR FORM AS FOLLOWS:
C               =  1  GENERALIZED EIGENVALUES WHOSE MAGNITUDE IS LESS
C                     THAN UNITY
C               =  0  ANY ORDER
C               = -1  GENERALIZED EIGENVALUES WHOSE REAL PARTS ARE
C                     LESS THAN ZERO;
C
C       IBAL    INTEGER
C               PARAMETER SPECIFYING THE BALANCING BEING EMPLOYED AS
```

```
C               FOLLOWS:
C               = 0  WARD BALANCING AND E IS A DESCRIPTOR MATRIX
C               = 1  CO-ORDINATE BALANCING AND E IS THE BALANCING
C                    TRANSFORMATION
C               = 2  NO BALANCING AND E IS A DESCRIPTOR MATRIX;
C
C       TYPE    LOGICAL
C               = .TRUE.  FOR CONTINUOUS-TIME SYSTEM
C               = .FALSE. FOR DISCRETE-TIME SYSTEM;
C
C       EFLAG   CHARACTER
C               FLAG SET TO 'Y' IF E IS A DESCRIPTOR MATRIX THAT IS
C               OTHER THAN THE IDENTITY MATRIX.
C
C     ON OUTPUT:
C
C       F       CONTAINS THE SOLUTION MATRIX X COMPUTED AS SHOWN
C               ABOVE;
C
C       Z       REAL(NRD,NN)
C               CONTAINS THE MATRIX PRODUCT
C                    ( E  0 )     ( Z11  Z12)
C                    (      )  *  (         )
C                    ( 0  I )     ( Z21  Z22)
C               WHERE Z IS THE ORTHOGONAL TRANSFORMATION MATRIX
C               DESCRIBED ABOVE;
C
C       ALFR    REAL(NN)
C               REAL PARTS OF THE DIAGONAL ELEMENTS THAT WOULD RESULT
C               IF THE Q AND Z TRANSFORMATIONS WERE APPLIED TO THE
C               G MATRIX SUCH THAT IT WOULD BE REDUCED COMPLETELY TO
C               TRIANGULAR FORM AND THE DIAGONAL ELEMENTS OF THE
C               TRANSFORMED F MATRIX (ALSO TRIANGULAR) WOULD BE REAL
C               AND POSITIVE;
C
C       ALFI    REAL(NN)
C               IMAGINARY PARTS OF THE DIAGONAL ELEMENTS THAT WOULD
C               RESULT IF THE Q AND THE Z TRANSFORMATIONS WERE
C               APPLIED TO THE G MATRIX SUCH THAT IT WOULD BE REDUCED
C               COMPLETELY TO TRIANGULAR FORM AND THE DIAGONAL
C               ELEMENTS OF THE TRANSFORMED F MATRIX (ALSO TRIANGULAR)
C               WOULD BE REAL AND POSITIVE.  NONZERO VALUES OCCUR IN
C               PAIRS; THE FIRST MEMBER IS POSITIVE AND THE SECOND
C               MEMBER IS NEGATIVE;
C
C       BETA    REAL(NN)
C               REAL NONNEGATIVE DIAGONAL ELEMENTS OF F THAT WOULD
C               RESULT IF G WERE REDUCED COMPLETELY TO TRAINGULAR
C               FORM; THE GENERALIZED EIGENVALUES ARE THEN GIVEN BY
```

```
C               THE RATIOS ((ALFR + I*ALFI)/BETA);
C
C       CPERM(1)
C               CONDITION ESTIMATE OF E*Z11 WITH RESPECT TO INVERSION;
C
C       IND(1)  ERROR FLAG AS FOLLOWS
C               = 0  INDICATES NORMAL RETURN
C               = NONZERO IF MORE THAT 50 ITERATIONS WERE REQUIRED TO
C               DETERMINE THE DIAGONAL BLOCKS FOR THE QUASITRIANGULAR
C               FORM;
C
C       IND(2)  ERROR FLAG AS FOLLOWS
C               = 0  INDICATES NORMAL RETURN
C               = 1  INDICATES ATTEMPTED REORDERING FAILED.
C     *****ALGORITHM NOTES:
C     NONE.
C
C     *****HISTORY:
C     THIS SUBROUTINE WAS WRITTEN BY W.F. ARNOLD, NAVAL WEAPONS
C     CENTER, CODE 35104, CHINA LAKE, CA  93555, AS PART OF THE
C     SOFTWARE PACKAGE RICPACK, SEPTEMBER 1983.
C
C     ------------------------------------------------------------
C
```

```
      SUBROUTINE NEWT(NR,NRR,NRW,NRX,N,M,E,A,B,CQC,R,S,RI,RSDM,X,W1,
     X                W2,W3,ALFR,ALFI,WK1,IPVT,RTOL,EFLAG,RFLAG,
     X                RDFLG,SFLAG,SEP,TYPE,MAXIT,INFO,NOUT)
C
C     *****PARAMETERS:
      INTEGER NR,NRR,NRW,NRX,N,M,IPVT(N),MAXIT,INFO,NOUT
      CHARACTER EFLAG,RFLAG,RDFLG,SFLAG
      DOUBLE PRECISION E(NR,N),A(NR,N),B(NR,M),CQC(NR,N),R(NR,M),
     X                 S(NR,M),RI(NR,M),RSDM(NRR,N),X(NRX,N),
     X                 W1(NRW,N),W2(NRW,N),W3(NRW,N),ALFR(N),
     X                 ALFI(N),WK1(N),RTOL,SEP
      LOGICAL TYPE
C
C     *****LOCAL VARIABLES:
      INTEGER       I,IER1,IER2,ITER,J
      DOUBLE PRECISION DPN1,EPS,E1N,H1N,TOL,T1,T2,T3,X1N
C
C     *****FORTRAN FUNCTIONS:
      DOUBLE PRECISION DSQRT
C
C     *****FUNCTION SUBPROGRAMS:
      DOUBLE PRECISION D1NRM
C
C     *****SUBROUTINES CALLED:
C     FBGAIN, LYPCND, LYPDSD, MADD, MMUL, MQF, MSCALE, MSUB, MULA,
C     SEPEST, TRNATB
C
C     ------------------------------------------------------------
C
C     *****PURPOSE:
C     GIVEN THE MODEL MATRICES FOR THE CONTINUOUS- OR DISCRETE-TIME
C     OPTIMAL REGULATOR PROBLEM THAT RESULTS IN A GENERALIZED
C     ALGEBRAIC RICCATI EQUATION (GARE), AND AN INITIAL GUESS FOR
C     THE SOLUTION TO THE GARE, THIS SUBROUTINE APPLIES A NEWTON
C     TYPE ITERATIVE REFINEMENT PROCEDURE.  TO GUARANTEE
C     CONVERGENCE, THE INITIAL GUESS MUST STABILIZE THE CLOSED LOOP
C     SYSTEM MATRIX E**-1*(A - B*K), WHERE
C
C          K = (R**-1)*(BT*X*E + ST)              CONTINUOUS
C
C          K = ((R + BT*X*B)**-1)*(BT*X*A + ST)   DISCRETE
C
C     AND T DENOTES MATRIX TRANSPOSE.
C
C     REF.:  ARNOLD, W.F., "ON THE NUMERICAL SOLUTION OF
C     ALGEBRAIC MATRIX RICCATI EQUATIONS," PHD THESIS, USC,
C     DECEMBER 1983.
C
C     *****PARAMETER DESCRIPTION:
```

```
C
C     ON INPUT:
C
C       NR      INTEGER
C               ROW DIMENSION OF THE ARRAYS CONTAINING THE MATRICES
C               E, A, B, CQC, R, S AND RI AS DECLARED IN THE MAIN
C               CALLING PROGRAM DIMENSION STATEMENT;
C
C       NRR     INTEGER
C               ROW DIMENSION OF THE ARRAY CONTAINING THE MATRIX
C               RSDM AS DECLARED IN THE MAIN CALLING PROGRAM
C               DIMENSION STATEMENT;
C
C       NRW     INTEGER
C               ROW DIMENSION OF THE ARRAYS CONTAINING THE MATRICES
C               W1, W2 AND W3 AS DECLARED IN THE MAIN CALLING
C               PROGRAM DIMENSION STATEMENT;
C
C       NRX     INTEGER
C               ROW DIMENSION OF THE ARRAY CONTAINING THE MATRIX X
C               AS DECLARED IN THE MAIN CALLING PROGRAM DIMENSION
C               STATEMENT;
C
C       N       INTEGER
C               ORDER OF THE SQUARE MATRICES E, A, CQC, RSDM AND X
C               ROW DIMENSION OF THE MATRICES B AND S;
C
C       M       INTEGER
C               ORDER OF THE SQUARE MATRICES R AND RI
C               COLUMN DIMENSION OF THE MATRICES B AND S;
C
C       E       REAL(NR,N)
C               MODEL DESCRIPTOR MATRIX;
C
C       A       REAL(NR,N)
C               MODEL SYSTEM MATRIX;
C
C       B       REAL(NR,M)
C               MODEL INPUT MATRIX;
C
C       CQC     REAL(NR,N)
C               MATRIX PRODUCT CT*Q*C WHERE T DENOTES MATRIX
C               TRANSPOSE;
C
C       R       REAL(NR,M)
C               CONTROL WEIGHTING MATRIX;
C
C       S       REAL(NR,M)
C               STATE - INPUT CROSS-WEIGHTING MATRIX;
```

```
C
C       RI      REAL(NR,M)
C               INVERSE OF THE CONTROL WEIGHTING MATRIX;
C
C       X       REAL(NRX,N)
C               INITIAL GUESS FOR RICCATI SOLUTION THAT MUST
C               STABILIZE THE CLOSED LOOP SYSTEM MATRIX;
C
C       W1,W2,W3
C               REAL(NRW,N)
C               SCRATCH ARRAYS OF SIZE AT LEAST N BY N;
C
C       WK1     REAL(N)
C               WORK VECTOR OF LENGTH AT LEAST N;
C
C       IPVT    INTEGER(N)
C               WORK VECTOR OF LENGTH AT LEAST N;
C
C       RTOL    REAL
C               TOLERANCE ON THE CONDITION ESTIMATE OF R+BT*X*B
C               WITH RESPECT TO INVERSION (DISCRETE PROBLEM).
C               ERROR RETURN IF THIS TOLERANCE IS EXCEEDED;
C
C       EFLAG   CHARACTER
C               FLAG SET TO 'Y' IF E IS OTHER THAN THE IDENTITY
C               MATRIX;
C
C       RFLAG   CHARACTER
C               FLAG SET TO 'Y' IF R IS OTHER THAN THE IDENTITY
C               MATRIX;
C
C       RDFLG   CHARACTER
C               FLAG SET TO 'Y' IF R IS A DIAGONAL MATRIX;
C
C       SFLAG   CHARACTER
C               FLAG SET TO 'Y' IF S IS OTHER THAN THE ZERO MATRIX;
C
C       TYPE    LOGICAL
C               = .TRUE.  FOR CONTINUOUS-TIME SYSTEM
C               = .FALSE. FOR DISCRETE-TIME SYSTEM;
C
C       MAXIT   INTEGER
C               MAXIMUM NUMBER OF NEWTON ITERATIONS PERMITTED;
C
C       NOUT    INTEGER
C               UNIT NUMBER OF OUTPUT DEVICE FOR ERROR WARNING
C               MESSAGES.
C
C     ON OUTPUT:
```

```
C
C       RSDM    REAL(NRR,N)
C               DIFFERENCE MATRIX BETWEEN SOLUTIONS AT LAST TWO
C               ITERATIONS;
C
C       X       REFINED RICCATI SOLUTION MATRIX;
C
C       ALFR    REAL(N)
C               REAL PARTS OF THE CLOSED LOOP SYSTEM EIGENVALUES
C               UNDER THE OPTIMAL FEEDBACK FOR THE RICCATI SOLUTION
C               AT THE LAST ITERATION;
C
C       ALFI    REAL(N)
C               IMAGINARY PARTS OF THE CLOSED LOOP SYSTEM
C               EIGENVALUES UNDER THE OPTIMAL FEEDBACK FOR THE
C               RICCATI SOLUTION AT THE LAST ITERATION;
C
C       WK1(1)  1-NORM OF THE RSDM MATRIX;
C
C       IPVT(1) CONVERGENCE CRITERIA INDICATOR;
C
C       SEP     REAL
C               ESTIMATE OF THE SEPARATION OF THE CLOSED LOOP
C               SPECTRUM AT THE LAST ITERATION (CONTINUOUS PROBLEM);
C
C       MAXIT   NUMBER OF ITERATIONS PERFORMED;
C
C       INFO    INTEGER
C               ERROR FLAG WITH THE FOLLOWING MEANINGS
C               = -1  NO CONVERGENCE
C               = -2  INITIAL GUESS NOT STABILIZING
C               = -3  E MATRIX NOT IDENTITY
C               = -4  INDICATES A FAILURE OF THE QR ALGORITHM TO
C                     DETERMINE THE EIGENVALUES IN SOLVING THE
C                     LYAPUNOV EQUATION
C               = -5  CONDITION ESTIMATE OF R+BT*X*B WITH RESPECT TO
C                     INVERSION EXCEEDS THE TOLERANCE VALUE RTOL.
C
C     *****ALGORITHM NOTES:
C     THE ALGORITHM CURRENTLY EMPLOYED IS BASED ON THE BARTELS-
C     STEWART ALGORITHM FOR LYAPUNOV EQUATIONS AND IS VALID FOR THE
C     CASE E = IDENTITY ONLY AT THIS TIME.
C
C     *****HISTORY:
C     THIS SUBROUTINE WAS WRITTEN BY W.F. ARNOLD, NAVAL WEAPONS
C     CENTER, CODE 35104, CHINA LAKE, CA  93555, AS PART OF THE
C     SOFTWARE PACKAGE RICPACK, SEPTEMBER 1983.
C
C     ------------------------------------------------------------
```

```
      SUBROUTINE ORDER (A,B,Z,NMAX,N,EPS,IFAIL,TYPE,IFIRST,IND)
C
C     *****PARAMETERS:
      INTEGER NMAX,N,IFAIL,IFIRST,IND(N)
      DOUBLE PRECISION A(NMAX,N),B(NMAX,N),Z(NMAX,N),EPS
      LOGICAL TYPE
C
C     *****LOCAL VARIABLES:
      INTEGER I,II,III,IS,ISTEP,K,L,LS,LS1,LS2,L1,L2,NUM
C
C     *****FORTRAN FUNCTIONS:
      DOUBLE PRECISION DABS
C
C     *****SUBROUTINES CALLED:
C     EXCHQZ
C
C     ------------------------------------------------------------
C
C     *****PURPOSE:
C     GIVEN THE UPPER TRIANGULAR MATRIX B AND UPPER HESSENBERG
C     MATRIX A WITH 1X1 OR 2X2 DIAGONAL BLOCKS, THIS SUBROUTINE
C     REORDERS THE DIAGONAL BLOCKS ALONG WITH THE GENERALIZED
C     EIGENVALUES CORRESPONDING TO THE REGULAR MATRIX PENCIL
C     A - LAMBDA*B  BY CONSTRUCTING ROW AND COLUMN EQUIVALENCE
C     TRANSFORMATIONS QT AND ZT.  THE COLUMN TRANSFORMATIONS ARE
C     THEN APPLIED TO THE MATRIX Z.
C     REF.:  VAN DOOREN, P., A GENERALIZED EIGENVALUE APPROACH FOR
C     SOLVING RICCATI EQUATIONS, SIAM J. SCI. STAT. COMPUT.,
C     VOL. 2, NO. 2, JUNE 1981, 121-135.
C
C     *****PARAMETER DESCRIPTION:
C
C     ON INPUT:
C
C       NMAX    INTEGER
C               ROW DIMENSION OF THE ARRAYS CONTAINING A,B,Z AS
C               DECLARED IN THE MAIN CALLING PROGRAM DIMENSION
C               STATEMENT;
C
C       N       INTEGER
C               ORDER OF THE MATRICES A,B,Z;
C
C       A       REAL(NMAX,N)
C               UPPER HESSENBERG MATRIX WITH 1X1 OR 2X2 DIAGONAL
C               BLOCKS.  ELEMENTS OUTSIDE THE UPPER HESSENBERG
C               STRUCTURE ARE ARBITRARY;
C
C       B       REAL(NMAX,N)
C               UPPER TRIANGULAR MATRIX.  ELEMENTS OUTSIDE THE
```

```
C                UPPER TRIANGULAR STRUCTURE ARE ARBITRARY;
C
C        EPS     REAL
C                REQUIRED ABSOLUTE ACCURACY OF THE RESULTS.
C                NORMALLY EQUAL TO THE MACHINE PRECISION;
C
C        TYPE    LOGICAL
C                CONTROL PARAMETER THAT SPECIFIES THE REGIONS OF THE
C                THE COMPLEX PLANE THAT THE GENERALIZED EIGENVALUES
C                ARE ORDERED BY.  TO CONTROL THE REGION THAT APPEARS
C                FIRST, SEE IFIRST BELOW
C                = .TRUE.  GENERALIZED EIGENVALUES ARE ORDERED BY
C                     REGION INSIDE THE COMPLEX LEFT HALF PLANE OR
C                     OUTSIDE THIS REGION
C                = .FALSE.  GENERALIZED EIGENVALUES ORDERED BY REGION
C                     INSIDE THE UNIT CIRCLE OR OUTSIDE THIS REGION;
C
C        IFIRST  INTEGER
C                CONTROL PARAMETER THAT SPECIFIES WHICH OF THE
C                REGIONS SPECIFIED BY TYPE(SEE ABOVE) APPEARS FIRST
C                (I.E. IN THE UPPER LEFT NXN BLOCK)
C                = -1  INSIDE REGION APPEARS FIRST
C                = +1  OUTSIDE REGION APPEARS FIRST
C                IFIRST IS ALTERED BY THIS SUBROUTINE;
C
C        IND     INTEGER(N)
C                WORKING ARRAY THAT IS ALTERED BY THIS SUBROUTINE.
C
C     ON OUTPUT:
C
C        A,B     UPPER HESSENBERG MATRIX, UPPER TRIANGULAR MATRIX
C                REORDERED AS SPECIFIED BY TYPE AND IFIRST(ABOVE);
C
C        Z       REAL(NMAX,N)
C                THIS ARRAY IS OVERWRITTEN BY THE PRODUCT OF THE
C                CONTENTS OF THE ARRAY Z(UPON ENTRY INTO THIS
C                SUBROUTINE), AND THE COLUMN TRANSFORMATIONS ZT
C                (CALCULATED BY THIS SUBROUTINE);
C
C        IFAIL   INTEGER
C                ERROR FLAG
C                = 1  INDICATES ATTEMPTED REORDERING FAILED
C                = 0  NORMAL RETURN.
C
C     *****ALGORITHM NOTES:
C     NONE.
C
C     *****HISTORY:
C     ORIGINAL VERSION THAT SORTED BY UNIT CIRCLE REGION OF COMPLEX
```

```
C     PLANE WRITTEN BY P. VAN DOOREN("A GENERALIZED EIGENVALUE
C     APPROACH FOR SOLVING RICCATI EQUATIONS", INTERNAL REPORT
C     NA-80-02, DEPT. OF COMPUTER SCIENCE, STANFORD UNIVERSITY,
C     1980).  THIS VERSION MODIFIED BY W. F. ARNOLD(DEPT. OF
C     ELECTRICAL ENGINEERING - SYSTEMS, UNIV. OF SOUTHERN CALIF.,
C     LOS ANGELES, CA  90089) TO INCLUDE THE SORTING CONTROL
C     PARAMETER "TYPE", SEPT 1982.
C
C     ------------------------------------------------------------
C
```

```
      SUBROUTINE LYPCND(NF,NH,N,F,H,Z,WR,WI,WK,IER1,IER2)
C
C     *****PARAMETERS:
      INTEGER NF,NH,N,IER1,IER2
      DOUBLE PRECISION F(NF,N),H(NH,N),Z(NF,N),WR(N),WI(N),WK(N)
C
C     *****LOCAL VARIABLES:
      INTEGER LOW,IGH
C
C     *****SUBROUTINES CALLED:
C     ORTHES,ORTRAN,HQRORT,MQFWO(MULWOA),SYMSLV(LINEQ,DGECOM,DGESLM)
C     TRNATA
C
C     ------------------------------------------------------------
C
C     *****PURPOSE:
C     THIS SUBROUTINE SOLVES THE CONTINUOUS TIME LYAPUNOV EQUATION
C
C           T
C          F *X + X*F + H = 0.
C
C     BY THE BARTELS-STEWART ALGORITHM (SEE REF.(1)).
C
C     *****PARAMETER DESCRIPTION:
C
C     ON INPUT:
C
C        NF,NH            ROW DIMENSIONS OF THE ARRAYS CONTAINING F
C                         (AND Z),H, RESPECTIVELY, AS DECLARED IN
C                         MAIN CALLING PROGRAM DIMENSION STATEMENT;
C
C        N                ORDER OF THE MATRICES F AND H;
C
C        F                N X N (REAL) MATRIX;
C
C        H                N X N SYMMETRIC MATRIX;
C
C        IER1             INTEGER VARIABLE; NORMALLY SET IER1 TO 0;
C                         IF IER1 IS SET TO A NON-ZERO INTEGER, THE
C                         REDUCTION OF F TO REAL SCHUR FORM IS
C                         SKIPPED AND THE ARRAYS F AND Z ARE ASSUMED
C                         TO CONTAIN THE REAL SCHUR FORM AND
C                         ACCOMPANYING ORTHOGONAL MATRIX THUS
C                         PERMITTING MORE EFFICIENT SOLUTION OF
C                         SEVERAL EQUATIONS WITH DIFFERENT CONSTANT
C                         TERMS H;
C
C        IER2             INTEGER VARIABLE; NORMALLY SET IER2 TO 0;
C                         IF ONLY A REAL SCHUR FORM OF F AND
```

```
C                       ASSOCIATED ORTHOGONAL SIMILARITY MATRIX Z
C                       ARE DESIRED SET IER2 TO A NON-ZERO
C                       INTEGER.
C
C     ON OUTPUT:
C
C        H              N X N ARRAY CONTAINING THE (SYMMETRIC)
C                       SOLUTION X OF THE LYAPUNOV EQUATION;
C
C        F              N X N ARRAY CONTAINING IN ITS UPPER
C                       TRIANGLE AND FIRST SUBDIAGONAL A REAL
C                       SCHUR FORM OF F;
C
C        Z              N X N ARRAY CONTAINING, ON OUTPUT, THE
C                       ORTHOGONAL MATRIX THAT REDUCES F TO REAL
C                       SCHUR FORM;
C
C        WR             REAL SCRATCH VECTOR OF LENGTH N; ON OUTPUT
C                       (WR(I),I=1,N) CONTAINS THE REAL PARTS OF
C                       THE EIGENVALUES OF F AND THUS CAN BE USED
C                       TO TEST THE STABILITY OF F;
C
C        WI             REAL SCRATCH VECTOR OF LENGTH N; ON OUTPUT
C                       CONTAINS THE IMAGINARY PARTS OF THE
C                       EIGENVALUES OF F;
C
C        WK             REAL SCRATCH VECTOR OF LENGTH N;
C
C        IER1           =0 FOR NORMAL RETURN (IF =0 ON INPUT),
C                       =J IF THE J-TH EIGENVALUE HAS NOT BEEN
C                       DETERMINED IN THE QR ALGORITHM (IF =0 ON
C                       INPUT).
C
C     *****ALGORITHM NOTES:
C     IT IS ASSUMED THAT F HAS NO EIGENVALUES WHICH SUM TO ZERO (THIS
C     CAN BE CHECKED FROM THE ARRAY WR). THIS IS SUFFICIENT TO
C     GUARANTEE A UNIQUE SOLUTION.
C     IF, MOREOVER, F IS STABLE THEN X IS NONNEGATIVE DEFINITE.
C
C     *****REFERENCES:
C        (1)   BARTELS, R.H., AND G.W. STEWART, SOLUTION
C              OF THE MATRIX EQUATION AX + XB = C,
C              ALGORITHM 432, COMM. ACM, 15(1972),820-826.
C
C     *****HISTORY:
C     WRITTEN BY ALAN J. LAUB (DEP'T. OF EE-SYSTEMS, U. OF SOUTHERN
C     CALIF., LOS ANGELES, CA 90089, PH.: (213) 743-5535)  SEP. 1977
C     MOST RECENT VERSION: JUNE 28, 1982.
C
```

```
      SUBROUTINE LYPDSD(NF,NH,N,F,H,Z,WR,WI,WK,U,IDIM,IER1,IER2)
C
C     *****PARAMETERS:
      INTEGER NF,NH,N,IER1,IER2,IDIM(N)
      DOUBLE PRECISION F(NF,N),U(NF,N),H(NH,N),Z(NF,N),WR(N),WI(N),
     X                 WK(N)
C
C     *****LOCAL VARIABLES:
      INTEGER LOW,IGH,KIN,KOUT
C
C     *****SUBROUTINES CALLED:
C     ORTHES,ORTRAN,HQRORT,MQFWO(MULWOA),DSTSLV(LINEQ,DDCOMP,DSOLVE)
C     TRNATA
C
C     ------------------------------------------------------------
C
C     *****PURPOSE:
C     THIS SUBROUTINE SOLVES THE DISCRETE TIME LYAPUNOV EQUATION
C
C           T
C          F *X*F- X = H.
C
C     BY A MODIFICATION OF THE BARTELS-STEWART ALGORITHM (SEE REFS.
C     (1) AND (2)).
C
C     *****PARAMETER DESCRIPTION:
C     ON INPUT:
C
C                          (AND Z,U),AND H ,RESPECTIVELY, AS DECLARED
C                          IN THE CALLING PROGRAM DIMENSION STATEMENT;
C
C        N                 ORDER OF THE MATRICES F AND C;
C
C        F                 N X N (REAL) MATRIX;
C
C        H                 N X N SYMMETRIC MATRIX;
C
C        IER1              INTEGER VARIABLE; NORMALLY SET IER1 TO 0;
C                          IF IER1 IS SET TO A NON-ZERO INTEGER, THE
C                          REDUCTION OF F TO REAL SCHUR FORM IS,
C                          SKIPPED AND THE ARRAYS F AND Z ARE ASSUMED
C                          TO CONTAIN THE REAL SCHUR FORM AND
C                          ACCOMPANYING ORTHOGONAL MATRIX THUS
C                          PERMITTING MORE EFFICIENT SOLUTION OF
C                          SEVERAL EQUATIONS WITH DIFFERENT CONSTANT
C                          TERMS H;
C
C        IER2              INTEGER VARIABLE; NORMALLY SET IER2 TO 0;
C                          IF ONLY A REAL SCHUR FORM OF F AND
```

```
C                       ASSOCIATED ORTHOGONAL SIMILARITY MATRIX Z
C                       ARE DESIRED SET IER2 TO A NON-ZERO
C                       INTEGER.
C
C     ON OUTPUT:
C
C        H              N X N ARRAY CONTAINING THE (SYMMETRIC)
C                       SOLUTION X OF THE LYAPUNOV EQUATION;
C
C        F              N X N ARRAY CONTAINING IN ITS UPPER
C                       TRIANGLE AND FIRST SUBDIAGONAL A REAL
C                       SCHUR FORM OF F;
C
C        Z              N X N ARRAY CONTAINING, ON OUTPUT, THE
C                       ORTHOGONAL MATRIX THAT REDUCES F TO REAL
C                       SCHUR FORM;
C
C        WR             REAL SCRATCH VECTOR OF LENGTH N; ON OUTPUT
C                       (WR(I),I=1,N) CONTAINS THE REAL PARTS OF
C                       THE EIGENVALUES OF F AND THUS CAN BE USED
C                       TO TEST THE STABILITY OF F;
C
C        WI             REAL SCRATCH VECTOR OF LENGTH N; ON OUTPUT
C                       CONTAINS THE IMAGINARY PART OF THE
C                       EIGENVALUES OF F;
C
C        WK             REAL SCRATCH VECTOR OF LENGTH N;
C
C        U              N X N REAL SCRATCH ARRAY;
C
C        IDIM           INTEGER SCRATCH VECTOR OF LENGTH N;
C
C        IER1           =0 FOR NORMAL RETURN (IF =0 ON INPUT),
C                       =J IF THE J-TH EIGENVALUE HAS NOT BEEN
C                       DETERMINED IN THE QR ALGORITHM (IF =0 ON
C                       INPUT).
C
C     *****ALGORITHM NOTES:
C     IT IS ASSUMED THAT F HAS NO EIGENVALUES WITH PRODUCT EQUAL TO
C     ZERO (THIS CAN BE CHECKED FROM THE ARRAY WR). THIS IS
C     SUFFICIENT TO GUARANTEE A UNIQUE SOLUTION.
C     IF, MOREOVER, F IS STABLE THEN X IS NONNEGATIVE DEFINITE.
C
C     *****REFERENCES:
C        (1)    BARTELS, R.H., AND G.W. STEWART, SOLUTION
C               OF THE MATRIX EQUATION AX + XB = C,
C               ALGORITHM 432, COMM. ACM, 15(1972),820-826.
C                                                          T
C        (2)    BARRAUD,A.Y., A NUMERICAL ALGORITHM TO SOLVE A XA-X=Q,
```

```
C               IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. AC-22,
C               NO.5, OCTOBER 1977,883-885.
C
C      *****HISTORY:
C      WRITTEN BY J.A.K. CARRIG (ELEC. SYS. LAB., M.I.T., RM. 35-427,
C      CAMBRIDGE, MA 02139, PH.: (617) 653-7263, SEPTEMBER 1978.
C      MOST RECENT VERSION: SEPT. 20, 1978.
C
C      ------------------------------------------------------------
C
```

```
      SUBROUTINE SEPEST(NR,N,T,Q,SEP,INFO)
C
C     *****PARAMETERS:
      INTEGER NR,N,INFO
      DOUBLE PRECISION T(NR,N),Q(NR,N),SEP
C
C     *****LOCAL VARIABLES:
      INTEGER IPVT(4),I,II,IM1,J,JP1,K1,K2,K1M1,L1,L2,L1M1,L1MK1,
     X        M,ND,NM1
      DOUBLE PRECISION A(4,4),VEC(4),Z(4),A1N,RCOND,S,TEMP,T1N
C
C     *****FORTRAN FUNCTIONS:
      DOUBLE PRECISION DABS,DMAX1,DMIN1,DSIGN
C
C     *****FUNCTION SUBPROGRAMS:
      DOUBLE PRECISION D1NRM
C
C     *****SUBROUTINES CALLED:
C     DGECOM, DGESLM, MSCALE, SYMSLV
C
C     ------------------------------------------------------------
C
C     *****PURPOSE:
C     GIVEN A QUASITRIANGULAR MATRIX T AND A SYMMETRIC MATRIX Q,
C     THIS SUBROUTINE COMPUTES
C
C       SEP = MIN( 1-NORM( T-TRANSPOSE*Q + Q*T)/ 1-NORM(Q))
C
C     REF.:  BARTELS, R.H. AND G.W. STEWART, "SOLUTION OF THE MATRIX
C     EQUATION A*X + X*B = C," COMM. OF THE ACM, VOL. 15,
C     PP. 820-826, 1972.
C     CLINE, A.K., MOLER, C.B., STEWART, G.W. AND J.H. WILKINSON,
C     "AN ESTIMATE OF THE CONDITION NUMBER OF A MATRIX," SIAM J. OF
C     NUMERICAL ANALYSIS, VOL. 16, PP. 368-375, 1979.
C
C     *****PARAMETER DESCRIPTION:
C
C     ON INPUT:
C
C       NR      INTEGER
C               ROW DIMENSION OF THE ARRAYS CONTAINING THE MATRICES
C               T AND Q AS DECLARED IN THE MAIN CALLING PROGRAM
C               DIMENSION STATEMENT;
C
C       N       INTEGER
C               ORDER OF THE SQUARE MATRICES Q AND T;
C
C       T       REAL(NR,N)
C               QUASITRIANGULAR INPUT MATRIX.
```

```
C
C     ON OUTPUT:
C
C       SEP     REAL
C               AN ESTIMATE OF THE QUANTITY SPECIFIED ABOVE;
C
C       Q       REAL(NR,N)
C               THE MATRIX THAT MINIMIZES THE QUANTITY SEP;
C
C       INFO    INTEGER
C               ERROR FLAG WITH THE FOLLOWING MEANINGS
C               = 0 INDICATES NORMAL RETURN
C               = (N+1)*L+K INDICATES THE L-TH AND K-TH EIGENVALUES
C                 OF T FORM A +/- PAIR, SO SEP IS EQUAL TO ZERO.
C
C     *****ALGORITHM NOTES:
C                                T
C     LET PHI(Y) = T*Y + Y*T .  Q IS OBTAINED BY INVERSE ITERATION ON
C           T
C     PHI*PHI .  THE STARTING VALUE OF Q IS CHOSEN AS FOLLOWS:
C     PARTITION ALL MATRICES CONFORMALLY WITH T.  Q IS CHOSEN TO
C     SATISFY
C                  T              T
C        T*Y + Y*T  = B,      T *Q + Q*T = Y/1-NORM(Y).
C
C     *****HISTORY:
C     THIS SUBROUTINE IS A MODIFIED VERSION OF THE SUBROUTINE OF THE
C     SAME NAME WRITTEN BY RALPH BEYERS, 2/82 REF.:  BEYERS, R.,
C     "HAMILTONIAN AND SYMPLECTIC ALGORITHMS FOR THE ALGEBRAIC
C     RICCATI EQUATION," PHD THESIS, CORNELL UNIVERSITY, PP.289-295,
C     JANUARY 1983.  THE MODIFICATIONS WERE MADE BY W.F. ARNOLD,
C     NAVAL WEAPONS CENTER, CODE 35104, CHINA LAKE, CA 93555, AS
C     PART OF THE SOFTWARE PACKAGE RICPACK, SEPTEMBER 1983.
C
C     ------------------------------------------------------------
C
```

```
      SUBROUTINE FBGAIN (NR,NRX,NRW,N,M,A,B,E,R,RI,S,X,FB,W,WK,IPVT,
     X                   EFLAG,RDFLG,RFLAG,SFLAG,TYPE)
C
C     *****PARAMETERS:
      INTEGER NR,NRX,NRW,N,M,IPVT(N)
      CHARACTER EFLAG,RDFLG,RFLAG,SFLAG
      DOUBLE PRECISION A(NR,N),B(NR,M),E(NR,N),R(NR,M),RI(NR,M),
     X    S(NR,M),X(NRX,N),FB(NRW,N),W(NRW,N),WK(N)
      LOGICAL TYPE
C
C     *****LOCAL VARIABLES:
      INTEGER I,J
      DOUBLE PRECISION COND
C
C     *****FORTRAN FUNCTIONS:
C     NONE.
C
C     *****SUBROUTINES CALLED:
C     MADD, MLINEQ, MMUL, MULA, MULB, TRNATA, TRNATB
C
C     ------------------------------------------------------------
C
C     *****PURPOSE:
C     GIVEN THE RICCATI SOLUTION AND THE MODEL MATRICES OF THE
C     OPTIMAL CONTROL PROBLEM, THIS SUBROUTINE CALCULATES THE
C     OPTIMAL FEEDBACK GAIN MATRIX FOR THE GENERALIZED CONTINUOUS-
C     OR DISCRETE-TIME OPTIMAL CONTROL PROBLEM.
C
C       CONTINUOUS:  FB = RI*(BT*X*E + ST)
C
C          DISCRETE:  FB = ((R + BT*X*B)**-1)*(BT*X*A + ST)
C
C     WHERE T DENOTES THE MATRIX TRANSPOSE.
C
C     REF.:  ARNOLD, W.F., "ON THE NUMERICAL SOLUTION OF
C     ALGEBRAIC MATRIX RICCATI EQUATIONS," PHD THESIS, USC,
C     DECEMBER 1983.
C
C     *****PARAMETER DESCRIPTION:
C
C     ON INPUT:
C
C       NR        INTEGER
C                 ROW DIMENSION OF THE ARRAYS CONTAINING
C                 A, B, E, R, RI AND S AS DECLARED IN THE MAIN
C                 CALLING PROGRAM DIMENSION STATEMENT;
C
C       NRX       INTEGER
C                 ROW DIMENSION OF THE ARRAY CONTAINING X AS DECLARED
```

```
C                IN THE MAIN CALLING PROGRAM DIMENSION STATEMENT;
C
C        NRW     INTEGER
C                ROW DIMENSION OF THE ARRAYS CONTAINING FB AND W AS
C                DECLARED IN THE MAIN PROGRAM DIMENSION STATEMENT;
C
C        N       INTEGER
C                ORDER OF THE SQUARE MATRICES A, E, AND X
C                ROW DIMENSION OF THE MATRICES B, S, AND FB;
C
C        M       INTEGER
C                ORDER OF THE SQUARE MATRICES R AND RI
C                COLUMN DIMENSION OF THE MATRICES B AND S;
C
C        A       REAL(NR,N)
C                MODEL SYSTEM MATRIX;
C
C        B       REAL(NR,M)
C                MODEL INPUT MATRIX;
C
C        E       REAL(NR,N)
C                MODEL DESCRIPTOR MATRIX;
C
C        R       REAL(NR,M)
C                INPUT WEIGHTING MATRIX;
C
C        RI      REAL(NR,M)
C                INVERSE OF THE INPUT WEIGHTING MATRIX;
C
C        S       REAL(NR,M)
C                STATE - INPUT CROSS-WEIGHTING MATRIX;
C
C        X       REAL(NRX,N)
C                ALGEBRAIC RICCATI EQUATION SOLUTION MATRIX;
C
C        W       REAL(NRW,N)
C                SCRATCH ARRAY OF SIZE AT LEAST N BY N;
C
C        WK      REAL(N)
C                WORKING VECTOR OF LENGTH AT LEAST N;
C
C        IPVT    INTEGER(M)
C                WORKING VECTOR OF LENGTH AT LEAST M;
C
C        EFLAG   CHARACTER
C                FLAG SET TO 'Y' IF E IS OTHER THAN THE IDENTITY
C                MATRIX;
C
C        RDFLG   CHARACTER
```

```
C                FLAG SET TO 'Y' IF R IS A DIAGONAL MATRIX;
C
C       RFLAG    CHARACTER
C                FLAG SET TO 'Y' IF R IS OTHER THAN THE IDENTITY
C                MATRIX;
C
C       SFLAG    CHARACTER
C                FLAG SET TO 'Y' IF S IS OTHER THAN THE ZERO MATRIX;
C
C       TYPE     LOGICAL
C                = .TRUE.  FOR CONTINUOUS-TIME SYSTEM
C                = .FALSE.  FOR DISCRETE-TIME SYSTEM.
C
C     ON OUTPUT:
C
C       FB       REAL(NRW,N)
C                OPTIMAL FEEDBACK GAIN MATRIX AS DESCRIBED ABOVE;
C
C       WK(1)    ESTIMATED CONDITION NUMBER OF R+BT*X*B WITH RESPECT
C                TO INVERSION (DISCRETE PROBLEM).
C
C     *****ALGORITHM NOTES:
C     NONE.
C
C     *****HISTORY:
C     THIS SUBROUTINE WAS WRITTEN BY W.F. ARNOLD, NAVAL WEAPONS
C     CENTER, CODE 35104, CHINA LAKE, CA  93555, AS PART OF THE
C     SOFTWARE PACKAGE RICPACK, SEPTEMBER 1983.
C
C     ------------------------------------------------------------
C
```

```
      SUBROUTINE CMPRS(NR,NRD,NRT,N,NN,NNPM,M,E,A,B,CQC,R,S,G,F,U,
     X                 WK,WK1,WK2,WK3,EFLAG,SFLAG,IBAL,INFO)
C
C     *****PARAMETERS:
      INTEGER NR,NRD,NRT,N,NN,NNPM,M,IBAL,INFO
      CHARACTER EFLAG,SFLAG
      DOUBLE PRECISION E(NR,N),A(NR,N),B(NR,M),CQC(NR,N),
     X    R(NR,M),S(NR,M),G(NRD,NN),F(NRD,NN),U(NRT,NNPM),
     X    WK(NRT,M),WK1(M),WK2(M),WK3(NNPM)
C
C     *****LOCAL VARIABLES:
      INTEGER I,J,JOB,K,NNPI,NPI,NPJ,NPK
C
C     *****FORTRAN FUNCTIONS:
C     NONE.
C
C     *****SUBROUTINES CALLED:
C     DSVDC
C
C     ------------------------------------------------------------
C
C     *****PURPOSE:
C     THIS SUBROUTINE EMPLOYS THE SINGULAR VALUE DECOMPOSITION TO
C     DETERMINE AN ORTHOGONAL MATRIX U, (2N+M) BY (2N+M), SUCH THAT
C
C
C               (     (     )  ( B )     ( 0 )
C               ( U11 ( U12 )  (   )     (   )
C               (     (     )  (-S )  =  ( 0 )
C               (-----(-----)  (---)     (---)
C               ( U21 ( U22 )  ( R )     ( RB)
C
C     AND THEN FORMS THE MATRIX PENCIL
C
C                    ( E  0 )     (     (  A    0 )               )
C        LAMBDA*U11*(       )  -  ( U11*(         ) + U12*(ST BT) )
C                    ( 0 ET )     (     (-CQC -AT )               )
C
C          =:  LAMBDA * F - G
C
C     WHERE T DENOTES MATRIX TRANSPOSE.
C     THIS PENCIL CAN THEN BE USED FOR SOLVING THE CONTINUOUS-TIME
C     GARE.
C
C     REF.:  ARNOLD, W.F., "ON THE NUMERICAL SOLUTION OF
C     ALGEBRAIC MATRIX RICCATI EQUATIONS," PHD THESIS, USC,
C     DECEMBER 1983.
C
C     *****PARAMETER DESCRIPTION:
```

```
C
C     ON INPUT:
C
C       NR       INTEGER
C                ROW DIMENSION OF THE ARRAYS CONTAINING THE MATRICES
C                E, A, B, CQC, R AND S AS DECLARED IN THE MAIN
C                CALLING PROGRAM DIMENSION STATEMENT;
C
C       NRD      INTEGER
C                ROW DIMENSION OF THE ARRAYS CONTAINING THE MATRICES
C                G AND F AS DECLARED IN THE MAIN CALLING PROGRAM
C                DIMENSION STATEMENT;
C
C       NRT      INTEGER
C                ROW DIMENSION OF THE ARRAYS CONTAINING THE MATRICES
C                U AND WK AS DECLARED IN THE MAIN CALLING PROGRAM
C                DIMENSION STATEMENT;
C
C       N        INTEGER
C                ORDER OF THE SQUARE MATRICES E, A AND CQC
C                ROW DIMENSION OF THE MATRICES B AND S;
C
C       NN       INTEGER
C                ORDER OF THE SQUARE MATRICES G AND F;
C
C       NNPM     INTEGER
C                = NN + M;
C
C       M        INTEGER
C                ORDER OF THE SQUARE MATRIX R
C                COLUMN DIMENSION OF THE MATRICES B AND S;
C
C       E        REAL(NR,N)
C                MODEL DESCRIPTOR MATRIX;
C
C       A        REAL(NR,N)
C                MODEL SYSTEM MATRIX;
C
C       B        REAL(NR,M)
C                MODEL INPUT MATRIX;
C
C       CQC      REAL(NR,N)
C                MATRIX PRODUCT CT*Q*C WHERE T DENOTES MATRIX
C                TRANSPOSE;
C
C       R        REAL(NR,M)
C                CONTROL WEIGHTING MATRIX;
C
C       S        REAL(NR,M)
```

```
C                STATE - INPUT CROSS-WEIGHTING MATRIX;
C
C       WK       REAL(NRT,M)
C                SCRATCH ARRAY OF SIZE AT LEAST (NN+M) BY M;
C
C       WK1,WK2  REAL(M)
C                WORKING VECTORS OF SIZE AT LEAST M;
C
C       WK3      REAL(NNPM)
C                WORKING VECTOR OF SIZE AT LEAST NN+M;
C
C       EFLAG    CHARACTER
C                FLAG SET TO 'Y' IF E IS OTHER THAN THE IDENTITY
C                MATRIX;
C
C       SFLAG    CHARACTER
C                FLAG SET TO 'Y' IF S IS OTHER THAN THE ZERO MATRIX;
C
C       IBAL     INTEGER
C                PARAMETER SET TO 1 IF CO-ORDINATE BALANCING IS
C                BEING USED.
C
C     ON OUTPUT:
C
C       G        REAL(NRD,NN)
C                MATRIX OF THE COMPRESSED PENCIL AS DEFINED ABOVE;
C
C       F        REAL(NRD,NN)
C                MATRIX OF THE COMPRESSED PENCIL AS DEFINED ABOVE;
C
C       U        REAL(NRT,NNPM)
C                ORTHOGONAL COMPRESSION MATRIX AS DEFINED ABOVE;
C
C       INFO     INTEGER
C                ERROR FLAG WITH MEANING AS FOLLOWS
C                = 0  NORMAL RETURN
C                = NONZERO  IF SINGULAR VALUE DECOMPOSITION COULD
C                           NOT BE CALCULATED.
C
C     *****ALGORITHM NOTES:
C     NONE.
C
C     *****HISTORY:
C     THIS SUBROUTINE WAS WRITTEN BY W.F. ARNOLD, NAVAL WEAPONS
C     CENTER, CODE 35104, CHINA LAKE, CA  93555, AS PART OF THE
C     SOFTWARE PACKAGE RICPACK, SEPTEMBER 1983.
C
C     ------------------------------------------------------------
C
```

```
      SUBROUTINE RINV(NR,NRD,N,NN,M,E,A,B,CQC,RI,G,F,WK1,WRK,RDFLG,
     X                RFLAG,EFLAG,IBAL,TYPE)
C
C     *****PARAMETERS:
      INTEGER NR,NRD,N,NN,M,IBAL
      CHARACTER RDFLG,RFLAG,EFLAG
      DOUBLE PRECISION E(NR,N),A(NR,N),B(NR,M),CQC(NR,N),RI(NR,M),
     X    G(NRD,NN),F(NRD,NN),WK1(NR,N),WRK(N)
      LOGICAL TYPE
C
C     *****LOCAL VARIABLES:
      INTEGER I,J,K,NPI,NPJ
C
C     *****FORTRAN FUNCTIONS:
C     NONE.
C
C     *****SUBROUTINES CALLED:
C     MULB, TRNATB
C
C     ---------------------------------------------------------------
C
C     *****PURPOSE:
C     THIS SUBROUTINE FORMS THE MATRIX PENCIL:
C
C                 ( E   0 )     (   A   -B*RI*BT )
C          LAMBDA*(       ) -  (                 )
C                 ( 0  ET )     ( -CQC      -AT  )
C
C      =:  LAMBDA * F  -  G
C
C     WHERE T DENOTES MATRIX TRANSPOSE.
C     THIS SUBROUTINE IS USEFUL IN SOLVING THE CONTINUOUS-TIME GARE.
C
C     REF.:  ARNOLD, W.F., "ON THE NUMERICAL SOLUTION OF
C     ALGEBRAIC MATRIX RICCATI EQUATIONS," PHD THESIS, USC,
C     DECEMBER 1983.
C
C     *****PARAMETER DESCRIPTION:
C
C     ON INPUT:
C
C       NR       INTEGER
C                ROW DIMENSION OF THE ARRAYS CONTAINING THE MATRICES
C                E, A, B, CQC, RI AND WK1 AS DECLARED IN THE MAIN
C                CALLING PROGRAM DIMENSION STATEMENT;
C
C       NRD      INTEGER
C                ROW DIMENSION OF THE ARRAYS CONTAINING THE MATRICES
C                F AND G AS DECLARED IN THE MAIN CALLING PROGRAM
```

```
C               DIMENSION STATEMENT;
C
C       N       INTEGER
C               ORDER OF THE SQUARE MATRICES E, A AND CQC
C               ROW DIMENSION OF THE MATRIX B;
C
C       NN      INTEGER
C               SIZE OF THE MATRIX PENCIL;
C
C       M       INTEGER
C               ORDER OF THE SQUARE MATRIX RI
C               COLUMN DIMENSION OF THE MATRIX B;
C
C       E       REAL(NR,N)
C               MODEL DESCRIPTOR MATRIX;
C
C       A       REAL(NR,N)
C               = A - B*RI*ST IN THE GENERALIZED CASE;
C
C       B       REAL(NR,M)
C               MODEL INPUT MATRIX;
C
C       CQC     REAL(NR,N)
C               =  CT*Q*C - S*RI*ST IN THE GENERALIZED CASE;
C
C       RI      REAL(NR,M)
C               INVERSE OF THE CONTROL WEIGHTING MATRIX;
C
C       WK1     REAL(NR,N)
C               SCRATCH ARRAY OF SIZE AT LEAST M BY N;
C
C       WRK     REAL(N)
C               WORKING VECTOR OF SIZE AT LEAST N;
C
C       RDFLG   CHARACTER
C               FLAG SET TO 'Y' IF RI IS A DIAGONAL MATRIX;
C
C       RFLAG   CHARACTER
C               FLAG SET TO 'Y' IF RI IS OTHER THAN THE IDENTITY
C               MATRIX;
C
C       EFLAG   CHARACTER
C               FLAG SET TO 'Y' IF E IS OTHER THAN THE IDENTITY
C               MATRIX;
C
C       IBAL    INTEGER
C               = 1  IF CO-ORDINATE BALANCING IS BEING USED;
C
C       TYPE    LOGICAL
```

```
C               = .TRUE.  FOR CONTINUOUS-TIME SYSTEM
C               = .FALSE. FOR DISCRETE-TIME SYSTEM.
C
C     ON OUTPUT:
C
C       G       REAL(NRD,NN)
C               PENCIL MATRIX AS DEFINED ABOVE;
C
C       F       REAL(NRD,NN)
C               PENCIL MATRIX AS DEFINED ABOVE.
C
C     *****ALGORITHM NOTES:
C     NONE.
C
C     *****HISTORY:
C     THIS SUBROUTINE WAS WRITTEN BY W.F. ARNOLD, NAVAL WEAPONS
C     CENTER, CODE 35104, CHINA LAKE, CA  93555, AS PART OF THE
C     SOFTWARE PACKAGE RICPACK, SEPTEMBER 1983.
C
C     ------------------------------------------------------------
C
```

```
      SUBROUTINE RESID(NR,NRR,NRW,NRX,N,M,E,A,B,CQC,R,S,RI,RSDM,X,W1,
     X                 W2,WK,IPVT,RTOL,EFLAG,RFLAG,RDFLG,SFLAG,RSD,
     X                 TYPE,NOUT)
C
C     *****PARAMETERS:
      INTEGER NR,NRR,NRW,NRX,N,M,IPVT(N),NOUT
      CHARACTER EFLAG,RFLAG,RDFLG,SFLAG
      DOUBLE PRECISION E(NR,N),A(NR,N),B(NR,M),CQC(NR,N),R(NR,M),
     X    S(NR,M),RI(NR,M),RSDM(NRR,N),X(NRX,N),W1(NRW,N),W2(NRW,N),
     X    WK(N),RTOL,RSD
      LOGICAL TYPE
C
C     *****LOCAL VARIABLES:
      INTEGER I
      DOUBLE PRECISION COND
C
C     *****FORTRAN FUNCTIONS:
C     NONE.
C
C     *****FUNCTION SUBPROGRAMS:
      DOUBLE PRECISION D1NRM
C
C     *****SUBROUTINES CALLED:
C     MADD, MLINEQ, MMUL, MQF, MSUB, MULA, TRNATB
C
C     ------------------------------------------------------------
C
C     *****PURPOSE:
C     THIS SUBROUTINE CALCULATES THE RESIDUAL MATRIX AND ITS 1-NORM
C     FOR THE GARE AS FOLLOWS:
C
C     CONTINUOUS:
C
C     RSDM = AT*X*E + ET*X*A - (BT*X*E+ST)T*RI*(BT*X*E+ST) + CQC
C
C
C     DISCRETE:
C
C     RSDM = AT*X*A - ET*X*E + CQC
C            - (BT*X*A+ST)T*((R+BT*X*B)**-1)*(BT*X*A+ST)
C
C
C     WHERE T DENOTES MATRIX TRANSPOSE.
C
C     REF.:  ARNOLD, W.F., "ON THE NUMERICAL SOLUTION OF
C     ALGEBRAIC MATRIX RICCATI EQUATIONS," PHD THESIS, USC,
C     DECEMBER 1983.
C
C     *****PARAMETER DESCRIPTION:
```

```
C
C     ON INPUT:
C
C       NR      INTEGER
C               ROW DIMENSION OF THE ARRAYS CONTAINING THE MATRICES
C               E, A, B, CQC, R, RI AND S AS DECLARED IN THE MAIN
C               CALLING PROGRAM DIMENSION STATEMENT;
C
C       NRR     INTEGER
C               ROW DIMENSION OF THE ARRAY CONTAINING THE MATRIX
C               RSDM AS DECLARED IN THE MAIN CALLING PROGRAM
C               DIMENSION STATEMENT;
C
C       NRW     INTEGER
C               ROW DIMENSION OF THE ARRAYS CONTAINING THE MATRICES
C               W1 AND W2 AS DECLARED IN THE MAIN CALLING PROGRAM
C               DIMENSION STATEMENT;
C
C       NRX     INTEGER
C               ROW DIMENSION OF THE ARRAY CONTAINING THE MATRIX X
C               AS DECLARED IN THE MAIN CALLING PROGRAM DIMENSION
C               STATEMENT;
C
C       N       INTEGER
C               ORDER OF THE SQUARE MATRICES E, A, CQC, RSDM AND X
C               ROW DIMENSION OF THE MATRICES B AND S;
C
C       M       INTEGER
C               ORDER OF THE SQUARE MATRICES R AND RI
C               COLUMN DIMENSION OF THE MATRICES B AND S;
C
C       E       REAL(NR,N)
C               MODEL DESCRIPTOR MATRIX;
C
C       A       REAL(NR,N)
C               MODEL SYSTEM MATRIX;
C
C       B       REAL(NR,M)
C               MODEL INPUT MATRIX;
C
C       CQC     REAL(NR,N)
C               MATRIX PRODUCT CT*Q*C WHERE T DENOTES MATRIX
C               TRANSPOSE;
C
C       R       REAL(NR,M)
C               CONTROL WEIGHTING MATRIX;
C
C       S       REAL(NR,M)
C               STATE - INPUT CROSS-WEIGHTING MATRIX;
```

```
C
C       RI      REAL(NR,M)
C               INVERSE OF THE CONTROL WEIGHTING MATRIX;
C
C       X       REAL(NRX,N)
C               SOLUTION MATRIX FOR THE GARE WHOSE RESIDUAL IS TO BE
C               DETERMINED;
C
C       W1,W2   REAL(NRW,N)
C               SCRATCH ARRAYS OF SIZE AT LEAST N BY N;
C
C       WK      REAL(N)
C               WORK VECTOR OF LENGTH AT LEAST N;
C
C       IPVT    INTEGER(N)
C               WORK VECTOR OF LENGTH AT LEAST N;
C
C       RTOL    REAL
C               TOLERANCE ON THE CONDITION ESTIMATE OF R+BT*X*B WITH
C               RESPECT TO INVERSION (DISCRETE PROBLEM).  IF THIS
C               TOLERANCE IS EXCEEDED AN ERROR MESSAGE IS PRINTED
C               AND AN ERROR RETURN IS MADE;
C
C       EFLAG   CHARACTER
C               FLAG SET TO 'Y' IF E IS OTHER THAN THE IDENTITY
C               MATRIX;
C
C       RFLAG   CHARACTER
C               FLAG SET TO 'Y' IF R IS OTHER THAN THE IDENTITY
C               MATRIX;
C
C       RDFLG   CHARACTER
C               FLAG SET TO 'Y' IF R IS A DIAGONAL MATRIX;
C
C       SFLAG   CHARACTER
C               FLAG SET TO 'Y' IF S IS OTHER THAN THE ZERO MATRIX;
C
C       TYPE    LOGICAL
C               = .TRUE.  FOR CONTINUOUS-TIME SYSTEM
C               = .FALSE. FOR DISCRETE-TIME SYSTEM;
C
C       NOUT    INTEGER
C               UNIT NUMBER OF OUTPUT DEVICE FOR ERROR WARNING
C               MESSAGES.
C
C     ON OUTPUT:
C
C       RSDM    REAL(NRR,N)
C               THE RESIDUAL MATRIX CALCULATED AS INDICATED ABOVE;
```

```
C
C       IPVT(1) ERROR FLAG AS FOLLOWS
C               = 0  NORMAL RETURN
C               = 1  THE CONDITION ESTIMATE OF R+BT*X*B WITH RESPECT
C                    TO INVERSION EXCEEDS THE TOLERANCE RTOL;
C
C       RSD     REAL
C               THE 1-NORM OF THE RESIDUAL MATRIX DIVIDED BY THE
C               1-NORM OF THE SOLUTION MATRIX.
C
C     *****ALGORITHM NOTES:
C     NONE.
C
C     *****HISTORY:
C     THIS SUBROUTINE WAS WRITTEN BY W.F. ARNOLD, NAVAL WEAPONS
C     CENTER, CODE 35104, CHINA LAKE, CA  93555, AS PART OF THE
C     SOFTWARE PACKAGE RICPACK, SEPTEMBER 1983.
C
C     ------------------------------------------------------------
C
```

```
      SUBROUTINE BALGEN (N,MA,A,MB,B,LOW,IGH,CSCALE,CPERM,WK)
C
C     *****PARAMETERS:
      INTEGER N,MA,MB,LOW,IGH
      DOUBLE PRECISION A(MA,N),B(MB,N),CSCALE(N),CPERM(N),WK(N,6)
C
C     *****LOCAL VARIABLES:
C     NONE.
C
C     *****FUNCTIONS:
C     NONE.
C
C     *****SUBROUTINES CALLED:
C     REDUCE, SCALEG, GRADEQ
C
C     ------------------------------------------------------------
C
C     *****PURPOSE:
C     THIS SUBROUTINE BALANCES THE MATRICES A AND B TO IMPROVE THE
C     ACCURACY OF COMPUTING THE EIGENSYSTEM OF THE GENERALIZED
C     EIGENPROBLEM A*X = (LAMBDA)*B*X.  THE ALGORITHM IS SPECIFI-
C     CALLY DESIGNED TO PRECEDE QZ TYPE ALGORITHMS, BUT IMPROVED
C     PERFORMANCE IS EXPECTED FROM MOST EIGENSYSTEM SOLVERS.
C     REF.:  WARD, R. C., BALANCING THE GENERALIZED EIGENVALUE
C     PROBLEM, SIAM J. SCI. STAT. COMPUT., VOL. 2, NO. 2, JUNE 1981,
C     141-152.
C
C     *****PARAMETER DESCRIPTION:
C
C     ON INPUT:
C
C       MA,MB   INTEGER
C               ROW DIMENSIONS OF THE ARRAYS CONTAINING MATRICES
C               A AND B RESPECTIVELY, AS DECLARED IN THE MAIN
C               CALLING PROGRAM DIMENSION STATEMENT;
C
C       N       INTEGER
C               ORDER OF THE MATRICES A AND B;
C
C       A       REAL(MA,N)
C               CONTAINS THE A MATRIX OF THE GENERALIZED
C               EIGENPROBLEM DEFINED ABOVE;
C
C       B       REAL(MB,N)
C               CONTAINS THE B MATRIX OF THE GENERALIZED
C               EIGENPROBLEM DEFINED ABOVE;
C
C       WK      REAL(N,6)
C               WORK ARRAY THAT MUST CONTAIN AT LEAST 6*N STORAGE
```

```
C               LOCATIONS.  WK IS ALTERED BY THIS SUBROUTINE.
C
C     ON OUTPUT:
C
C       A,B     CONTAIN THE BALANCED A AND B MATRICES;
C
C       LOW     INTEGER
C               BEGINNING INDEX OF THE SUBMATRICES OF A AND B
C               CONTAINING THE NON-ISOLATED EIGENVALUES;
C
C       IGH     INTEGER
C               ENDING INDEX OF THE SUBMATRICES OF A AND B
C               CONTAINING THE NON-ISOLATED EIGENVALUES.  IF
C               IGH = 1 (LOW = 1 ALSO), THE A AND B MATRICES HAVE
C               BEEN PERMUTED INTO UPPER TRIANGULAR FORM AND HAVE
C               NOT BEEN BALANCED;
C
C       CSCALE  REAL(N)
C               CONTAINS THE EXPONENTS OF THE COLUMN SCALING FACTORS
C               IN ITS LOW THROUGH IGH LOCATIONS AND THE REDUCING
C               COLUMN PERMUTATIONS IN ITS FIRST LOW-1 AND ITS
C               IGH+1 THROUGH N LOCATIONS;
C
C       CPERM   REAL(N)
C               CONTAINS THE COLUMN PERMUTATIONS APPLIED IN GRADING
C               THE A AND B SUBMATRICES IN ITS LOW THROUGH IGH
C               LOCATIONS;
C
C       WK      CONTAINS THE EXPONENTS OF THE ROW SCALING FACTORS
C               IN ITS LOW THROUGH IGH LOCATIONS, THE REDUCING ROW
C               PERMUTATIONS IN ITS FIRST LOW-1 AND ITS IGH+1
C               THROUGH N LOCATIONS, AND THE ROW PERMUTATIONS
C               APPLIED IN GRADING THE A AND B SUBMATRICES IN ITS
C               N+LOW THROUGH N+IGH LOCATIONS.
C
C     *****ALGORITHM NOTES:
C     NONE.
C
C     *****HISTORY:
C     WRITTEN BY R. C. WARD.......
C
C     ------------------------------------------------------------
C
```

```
      SUBROUTINE BALGBK (N,MZ,Z,M,LOW,IGH,CSCALE,CPERM)
C
C     *****PARAMETERS:
      INTEGER N,MZ,M,LOW,IGH
      DOUBLE PRECISION Z(MZ,N),CSCALE(N),CPERM(N)
C
C     *****LOCAL VARIABLES:
C     NONE.
C
C     *****FUNCTIONS:
C     NONE.
C
C     *****SUBROUTINES CALLED:
C     GRADBK, SCALBK
C
C     ------------------------------------------------------------
C
C     *****PURPOSE:
C     THIS SUBROUTINE BACK TRANSFORMS THE EIGENVECTORS OF A
C     GENERALIZED EIGENVALUE PROBLEM A*X = (LAMBDA)*B*X, THAT WAS
C     BALANCED BY SUBROUTINE BALGEN, TO THOSE OF THE ORIGINAL
C     PROBLEM.
C     REF.:  WARD, R. C., BALANCING THE GENERALIZED EIGENVALUE
C     PROBLEM, SIAM J. SCI. STAT. COMPUT., VOL. 2, NO. 2, JUNE 1981,
C     141-152.
C
C     *****PARAMETER DESCRIPTION:
C
C     ON INPUT:
C
C       MZ      INTEGER
C               ROW DIMENSION OF THE ARRAY Z AS SPECIFIED IN THE
C               MAIN CALLING PROGRAM DIMENSION STATEMENT;
C
C       N       INTEGER
C               ORDER OF THE MATRICES A AND B IN THE EIGENPROBLEM;
C
C       M       INTEGER
C               SPECIFIES THE NUMBER OF EIGENVECTORS TO BE TRANS-
C               FORMED;
C
C       Z       REAL(MZ,N)
C               CONTAINS THE EIGENVECTORS TO BE TRANSFORMED;
C
C       LOW     INTEGER
C               SPECIFIES THE BEGINNING INDEX OF THE SUBMATRICES OF
C               A AND B WHICH WERE BALANCED;
C
C       IGH     INTEGER
```

```
C               SPECIFIES THE ENDING INDEX OF THE SUBMATRICES OF
C               A AND B WHICH WERE BALANCED;
C
C       CSCALE  REAL(N)
C               CONTAINS THE REDUCING COLUMN PERMUTATIONS AND
C               SCALING INFORMATION AS RETURNED FROM BALGEN;
C
C       CPERM   REAL(N)
C               CONTAINS IN ITS LOW THROUGH IGH LOCATIONS THE
C               COLUMN PERMUTATIONS APPLIED IN GRADING THE A
C               AND B SUBMATRICES AS RETURNED FROM BALGEN.
C
C     ON OUTPUT:
C
C       Z       CONTAINS THE TRANSFORMED EIGENVECTORS.
C
C     *****ALGORITHM NOTES:
C     NONE.
C
C     *****HISTORY:
C     WRITTEN BY R. C. WARD.......
C
C     ------------------------------------------------------------
C
```

INITIAL DISTRIBUTION

2 Naval Air Systems Command (AIR-00D4)
3 Chief of Naval Research, Arlington (ONR-411-MA, Holland)
2 Naval Sea Systems Command (SEA-09B312)
1 Commander in Chief, U.S. Pacific Fleet (Code 325)
1 Commander, Third Fleet, Pearl Harbor
1 Commander, Seventh Fleet, San Francisco
3 Naval Ship Weapon Systems Engineering Station, Port Hueneme
  Code 5711, Repository (2)
  Code 5712 (1)
1 Naval War College, Newport
1 Air Force Institute of Technology, Wright-Patterson Air Force Base (C. Houpis)
12 Defense Technical Information Center
1 Langley Research Center (NASA), Hampton (E. Armstrong)
1 Alphatech, Inc., Burlington, MA (Dr. N. R. Sandell, Jr.)
1 Cornell University, Ithaca, NY (Department of Computer Science (C. Van Loan)
1 Honeywell, Inc., Systems and Research Center, Minneapolis, MN (Dr. N. A. Lehtomaki)
1 Integrated Systems Incorporated, Palo Alto, CA (Dr. R. A. Walker)
1 Litton Systems, Inc., Woodland Hills, CA (MS 67/35, Dr. R. W. Bass)
1 McGill University, Montreal, Canada (Department of Computer Science, C. Paige)
1 New Mexico Engineering Research Institute, Albuquerque, NM (Department of Computer Science, C. Moler)
1 Northern Illinois University, Dekalb, IL, (Department of Mathematical Sciences, R. Byers)
1 Rice University, Houston, TX (Department of Mathematical Sciences, J. E. Dennis, Jr.)
1 Systems Control Technology, Inc., Palo Alto, CA (Dr. A. Emami-Naeini)
1 University of California, Lawrence Livermore National Laboratory, Livermore, CA (Dr. C. J. Herget)
5 University of California, Santa Barbara, Santa Barbara, CA (Department of Electrical & Computer Engineering, A. J. Laub)
3 University of Southern California, Los Angeles, CA (Department of Electrical Engineering-Systems, L. M. Silverman)
1 University of Wisconsin, Madison, WI (Department of Mathematics, D. Russell)